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The Hawking effect for a collapsing star in an initial state of KMS type

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Abstract

We prove the Hawking effect for gravitational collapse of a charged star in an expanding universe or not, stationary in the past and collapsing to a black hole in the future. In the past, the quantum initial state of the Dirac fields was given by a KMS state with unspecified temperature. With the same physical model, this paper generalizes our previous work to the case of a quantum initial state of KMS type rather than a Boulware vacuum.

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1. Introduction

This paper extends our previous investigation [15] about a rigorous mathematical proof of the Hawking effect [11, 21] for the Dirac field. In [15] we considered a charged star, stationary in the past and collapsing to a black hole in the framework of the semiclassical approximation where the back-reaction of the field on the metric is neglected (i.e. the gravitational perturbations due to the particle are small). Hence a solution of the Einstein–Maxwell equations in vacuum enables us to calculate the Dirac equation in this curved space–time. Furthermore, the quantum state in the past was given by the Boulware vacuum that corresponds to the classical concept of vacuum for a static observer. Again in our previous work, the theorem about the Hawking effect was proved when the cosmological constant Λ is positive (de Sitter–Reissner–Nordstrøm space–time) rather than zero (Reissner–Nordstrøm space–time). But, in the case of a collapse in a de Sitter–Reissner–Nordstrøm universe $\Lambda > 0$, the physically relevant state in the past is the KMS state of temperature given by Gibbons–Hawking [10]. A KMS state corresponds to the Gibbs equilibrium state describing the thermodynamic models for a gas of noninteracting Fermi particles with a given temperature and chemical potential. This Gibbons–Hawking temperature is associated with the cosmological horizon and its surface gravity κ_+: $T_{\mathrm{GH}} = \frac{2\pi}{\kappa_+}$. In this new work and always for the semiclassical regime, we study the same collapsing star with $\Lambda \geqslant 0$, but in the past we set a quantum state of KMS type with unspecified temperature and chemical potential. Hence the previous open question about case $\Lambda > 0$ is covered by this study.

As in [15] and according to an observer at rest, we prove the emergence of a thermal state coming from the future black hole which is independent of the story of the collapse and the nature of the star surface. Moreover, with the result of this paper and the previous, we also remark that the choice of the initial state in the past does not modify the characteristic of the flux of particles coming from the horizon of the future black hole.

During the collapse, the star becomes a black hole. This black hole is described in terms of the Schwarzschild coordinates (t, r, ω) as the globally hyperbolic manifold $(\mathcal{M}_{\mathrm{bh}}, g)$ (see, e.g., [12, 16, 22])

$$\begin{aligned}
&\mathcal{M}_{\mathrm{bh}} := \mathbb{R}_t \times]r_0, r_+[_r \times S^2_\omega, \qquad 0 < r_0 < r_+ \leqslant +\infty,\\
&g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = F(r)\,\mathrm{d}t^2 - F^{-1}(r)\,\mathrm{d}r^2 - r^2\,\mathrm{d}\omega^2,\\
&\mathrm{d}\omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2, \qquad \omega = (\theta, \varphi) \in [0, \pi] \times [0, 2\pi],\\
&F(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\Lambda r^2}{3},
\end{aligned} \tag{1}$$

where $Q \in \mathbb{R}$, $M > 0$ and $\Lambda \geqslant 0$ are respectively the electric charge, the mass and the cosmological constant. Here r_0 and r_+ are the radius of the horizon of the black hole and the radius of the cosmological horizon, moreover

$$\begin{aligned}
&F(r_0) = F(r_+) = 0, \qquad 2\kappa_0 = F'(r_0) > 0,\\
&2\kappa_+ = F'(r_+) < 0, \qquad r \in]r_0, r_+[\Rightarrow F(r) > 0,
\end{aligned} \tag{2}$$

with κ_0, κ_+ the surface gravity at the black-hole horizon and at the cosmological horizon. If the cosmological constant $\Lambda = 0$, then $(\mathcal{M}_{\mathrm{bh}}, g)$ describes the asymptotically flat space–time of Reissner–Nordstrøm by

$$\begin{aligned}
&F(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}, \qquad 0 < |Q| \leqslant M,\\
&r_0 = M + \sqrt{M^2 - Q^2}, \qquad r_+ = +\infty.
\end{aligned}$$

We introduce the Regge–Wheeler coordinate such that

$$\frac{\mathrm{d}r_*(r)}{\mathrm{d}r} = F^{-1}. \tag{3}$$

With this new radial coordinate, the horizons are pushed away at infinity:

$$\begin{aligned}
&r_*(r) \to -\infty \iff r \to r_0, \qquad \Lambda \geqslant 0\\
&r_*(r) \to +\infty \iff r \to r_+, \qquad \Lambda > 0,\\
&r_*(r) \to +\infty \iff r \to +\infty, \qquad \Lambda = 0.
\end{aligned}$$

Hence, we define the space–time outside the collapsing star with mass $M > 0$ and r_*-radius $z(t)$, $t \in \mathbb{R}$ in an expanding or asymptotically flat universe, such that

$$\mathcal{M}_{\mathrm{coll}} := \left\{(t, r_*, \omega) \in \mathbb{R}_t \times \mathbb{R}_{r_*} \times S^2_\omega, \quad r_* \geqslant z(t)\right\}. \tag{4}$$

The reasonable assumptions of generic collapse examined in [1] lead to the following properties for $z(t)$:

$$z \in C^2(\mathbb{R}), \qquad \forall t \in \mathbb{R}, \qquad -1 < \dot{z}(t) \leqslant 0, \qquad t \leqslant 0 \quad \Rightarrow \quad z(t) = z(0) < 0, \tag{5}$$

$$z(t) = -t - C_{\kappa_0}\,\mathrm{e}^{-2\kappa_0 t} + \varpi(t), \qquad C_{\kappa_0} > 0, \qquad |\varpi(t)| + |\dot{\varpi}(t)| = \in O(\mathrm{e}^{-4\kappa_0 t}),$$
$$t \to +\infty. \tag{6}$$

According to the Birkhoff theorem, and since the spherical symmetry of the star is maintained during the collapse, the metric on $\mathcal{M}_{\rm coll}$ is just the Lorentzian metric g defined in (1).

Since we adopt the semiclassical approximation, on $(\mathcal{M}_{\rm coll}, g)$ we consider the Dirac equation for a fermion of mass $m > 0$ and charge $q \in \mathbb{R}$:

$$\mathrm{i}\gamma^{\mu}\bar{\nabla}_{\mu}\Psi + \mathrm{i}q\frac{Q}{r}\Psi - m\Psi = 0. \tag{7}$$

The term $\frac{Q}{r}$ is the electromagnetic potential since we take electromagnetic interactions between the field and the charged star into account. Here γ^{μ} are the Dirac matrices in curved space–time and $\bar{\nabla}_{\mu}$ the spinor fields' covariant derivative. Our model of the star is very simple and very convenient since our star is in fact a mirror. These assumptions enable us to avoid treating different interactions and behaviour of the fluid inside the star during the collapse. But as Hawking remarks in [11], the important blue shift arising in this phenomenon leads us to use the geometrical optics approximation in the last moments of the collapse. In this case, we can consider that the field would propagate through the star and out. Hence, the mirror model does not seem restrictive although the study with a more general star model is a very interesting problem which probably does not change the result. Therefore, on the star surface

$$\mathcal{S} := \left\{(t, r_*, \omega) \in \mathbb{R}_t \times \mathbb{R}_{r_*} \times S^2_{\omega}, \quad r_* = z(t)\right\},$$

we put the following conservative boundary condition, written for $(t, r_*, \omega) \in \mathcal{S}$, as

$$n_j\gamma^j\Psi(t, r_*, \omega) = \mathrm{i}\,\mathrm{e}^{\mathrm{i}\nu\gamma^5}\Psi(t, r_*, \omega), \qquad \gamma^5 := -\mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3 \tag{8}$$

where n_j is the outgoing normal of the subset of $\mathbb{R}_t \times \mathbb{R}_{r_*} \times S^2_{\omega}$ and ν the chiral angle. We suppose for technical reasons that $\nu \in \mathbb{R}$ if $r_+ < +\infty$, and $\nu \neq (2k+1)\pi$, $k \in \mathbb{Z}$ if $r_+ = +\infty$. This conservative boundary condition is the generalized *MIT bag* boundary condition [5] which causes a reflection of the fields on the star surface.

In section 2 of this paper, we state the theorem giving a solution of the mixed hyperbolic problems (7) and (8) with the help of a propagator. In this same section, we also introduce the useful wave operators outside the future black hole. In the fifth section, we state and interpret the main theorem of this work using the quantum field theory. To do this, we construct the local algebra of observables $\mathfrak{U}(\mathcal{M}_{\rm coll})$ as in [6] and use the wave operators of section 2. Finally in the last section, we expose the mathematical proof of the main theorem of this paper.

2. Classical fields

2.1. Dirac equation

By using definition (1) and a calculation from [2] and [17] for equation (7), we set the hyperbolic mixed problem in a Hamiltonian form on $(\mathcal{M}_{\rm coll}, g)$ related to (7) and (8):

$$\partial_t\Psi = \mathrm{i}\boldsymbol{D}_t\Psi, \qquad z(t) < r_*, \tag{9}$$

$$\frac{\dot{z}\gamma^0 - \gamma^1}{\sqrt{1-\dot{z}^2}}\Psi(t, z(t)) = \mathrm{i}\,\mathrm{e}^{\mathrm{i}\nu\gamma^5}\Psi(t, z(t)) \tag{10}$$

$$\Psi(t = s, .) = \Psi_s(.) \in \boldsymbol{L}^2_s, \tag{11}$$

where $\boldsymbol{L}^2_t$ is the energy space such that

$$\left(\boldsymbol{L}^2_t := L^2\left(]z(t), +\infty[_{r_*} \times S^2_{\omega}, r^2F^{1/2}(r)\,\mathrm{d}r_*\,\mathrm{d}\omega\right)^4, \|.\|_t\right) \tag{12}$$

and

$$D_t = -\frac{qQ}{r} + \Gamma^1\left(\partial_{r_*} + \frac{F(r)}{r} + \frac{F'(r)}{4}\right) + \sqrt{F(r)}\left(\frac{\Gamma^2}{r}\left(\partial_\theta + \frac{1}{2}\cot\theta\right) + \frac{\Gamma^3}{r\sin\theta}\partial_\varphi + \Gamma^4\right), \tag{13}$$

$$\Gamma^1 := \mathrm{i}\gamma^0\gamma^1 = \mathrm{i}\,\mathrm{Diag}(-1,1,1,-1), \qquad \Gamma^2 := \mathrm{i}\gamma^0\gamma^2, \qquad \Gamma^3 := \mathrm{i}\gamma^0\gamma^3, \qquad \Gamma^4 := -m\gamma^0, \tag{14}$$

with

$$\mathcal{D}(D_t) = \left\{\Psi \in L_t^2, D_t\Psi \in L_t^2; \frac{\dot{z}\gamma^0 - \gamma^1}{\sqrt{1-\dot{z}^2}}\Psi(z(t),\omega) = \mathrm{i}\,\mathrm{e}^{\mathrm{i}\nu\gamma^5}\Psi(z(t),\omega)\right\}. \tag{15}$$

Here the Dirac matrices, γ^k, satisfy

$$\gamma^a\gamma^b + \gamma^b\gamma^a = 2\eta^{ab}I_{\mathbb{R}^4}, \qquad a,b = 0,\ldots,3, \qquad \eta^{ab} = \mathrm{Diag}(1,-1,-1,-1). \tag{16}$$

$$\gamma^0 = \mathrm{i}\begin{pmatrix} 0 & \sigma^0 \\ -\sigma^0 & 0 \end{pmatrix}, \qquad \gamma^k = \mathrm{i}\begin{pmatrix} 0 & \sigma^k \\ \sigma^k & 0 \end{pmatrix}, \qquad k = 1,2,3, \tag{17}$$

with the Pauli matrices,

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^3 = \mathrm{i}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{18}$$

We introduce the following notation:

$$\forall\Phi \in L_t^2, \qquad \|\Phi\|_t = \|[\Phi]_L\|, \qquad [\Phi]_L(r_*,\omega) = \begin{cases} \Phi(r_*,\omega) & r_* \in \,]z(t),+\infty[_{r_*} \\ 0 & r_* \in \mathbb{R}\backslash]z(t),+\infty[_{r_*} \end{cases}.$$

According to proposition III.2 in [2], a unique solution $\Psi(t)$ of (9), (10) and (11) can be expressed with the propagator $U(t,s)$:

Proposition 2.1. *Given $\Psi_s \in \mathcal{D}(D_s)$, then there exists a unique solution $[\Psi(.)]_L = [U(.,s)\Psi_s]_L \in C^1(\mathbb{R}_t, L^2_{\mathrm{BH}})$ of (9), (10) and (11) such that, for all $t \in \mathbb{R}$*

$$\Psi(t) \in \mathcal{D}(D_s), \qquad \|\Psi(t)\|_t = \|\Psi_s\|_s.$$

Moreover, $U(t,s)$ can be extended in an isometric strongly continuous propagator from L_s^2 onto L_t^2.

In the same way, we consider the hyperbolic problem related to (7) on $(\mathcal{M}_{\mathrm{bh}}, g)$:

$$\partial_t\Psi = \mathrm{i}D_{\mathrm{BH}}\Psi \tag{19}$$

$$\Psi(t=0,.) = \Psi_{\mathrm{BH}}(.) \in L^2_{\mathrm{BH}}, \tag{20}$$

where the differential operator D_{BH} has form (13) but defined on

$$\left(L^2_{\mathrm{BH}} := L^2\left(\mathbb{R}_{r_*} \times S^2_\omega, r^2F^{1/2}(r)\,\mathrm{d}r_*\,\mathrm{d}\omega\right)^4, \|.\|\right). \tag{21}$$

In [13], we prove that D_{BH} is self-adjoint with dense domain

$$\mathcal{D}(D_{\mathrm{BH}}) = \left\{\Psi \in L^2_{\mathrm{BH}}, D_{\mathrm{BH}}\Psi \in L^2_{\mathrm{BH}}\right\}. \tag{22}$$

Hence by the spectral theorem, we have

Proposition 2.2. *Problems (19) and (20) have a unique solution $\Psi \in C^0(\mathbb{R}_t, L^2_{\mathrm{BH}})$ given by the strongly continuous unitary group $U(t) := \mathrm{e}^{\mathrm{i}tD_{\mathrm{BH}}}$:*

$$\Psi(t) = U(t)\Psi_{\mathrm{BH}}, \qquad \Psi(0) = \Psi_{\mathrm{BH}}.$$

Moreover

$$\|\Psi(t)\| = \|\Psi_{\mathrm{BH}}\|.$$

2.2. Scattering for Dirac fields by an eternal black hole

Our result on the Hawking effect follows from an asymptotic analysis for the propagator $\boldsymbol{U}(0,T)$ as $T\to+\infty$. As the star becomes a black hole as $T\to+\infty$, we strongly state that the dynamics are simpler in the vicinity of the following two asymptotic regions: $r_*\to-\infty$ (black-hole horizon) and $r_*\to+\infty$ (cosmological horizon when $\Lambda>0$ or the asymptotically flat space–time when $\Lambda=0$). This is the reason why we introduce the wave operators for the eternal charged black hole. The existence and the asymptotic completeness for these operators have already been the subject of two previous works: [13, 14]. To investigate the behaviour of the Dirac fields near the black-hole horizon (resp. cosmological horizon $\Lambda>0$ or asymptotically flat region $\Lambda=0$), we choose a cut function $\chi_\leftarrow\in C^\infty(\mathbb{R}_{r_*})$ (resp. $\chi_\rightarrow\in C^\infty(\mathbb{R}_{r_*})$) satisfying

$$\begin{aligned}&\exists\, a,b\in\mathbb{R},\qquad 0<a<b<1,\\ &\chi_\leftarrow(r_*)=\begin{cases}1 & r_*<a\\ 0 & r_*>b,\end{cases}\qquad(\text{resp. } \chi_\rightarrow=1-\chi_\leftarrow).\end{aligned}\tag{23}$$

As regards the asymptotic behaviour of the fields as $r_*\to-\infty$ (resp. $r_*\to+\infty$ when $\Lambda>0$), we compare the solution of (19) on $\boldsymbol{L}^2_{\mathrm{BH}}$ with the solution of

$$\partial_t\Psi_\leftarrow=\mathrm{i}\boldsymbol{D}_\leftarrow\Psi_\leftarrow\qquad(\text{resp. } \partial_t\Psi_\rightarrow=\boldsymbol{D}_{\Lambda,\rightarrow}\Psi_\rightarrow)\tag{24}$$

where

$$\boldsymbol{D}_\leftarrow:=\Gamma^1\partial_{r_*}-\frac{qQ}{r_0}\qquad\left(\text{resp. } \boldsymbol{D}_{\Lambda,\rightarrow}:=\Gamma^1\partial_{r_*}-\frac{qQ}{r_+}\right)$$

is self-adjoint on

$$\boldsymbol{L}^2_\leftarrow:=L^2\big(\mathbb{R}_{r_*}\times S^2_\omega\,\mathrm{d}r_*\,\mathrm{d}\omega\big)^4,\qquad\big(\text{resp. } \boldsymbol{L}^2_{\Lambda,\rightarrow}:=\boldsymbol{L}^2_\leftarrow,\quad\Lambda>0\big),$$

with the dense domain

$$\mathcal{D}(\boldsymbol{D}_\leftarrow)=H^1\big(\mathbb{R}_{r_*};L^2\big(S^2_\omega\big)\big)^4\qquad\big(\text{resp. } \mathcal{D}(\boldsymbol{D}_{\Lambda,\rightarrow})=H^1\big(\mathbb{R}_{r_*};L^2\big(S^2_\omega\big)\big)^4\big).$$

Since Γ^1 is diagonal, we remark that equations (24) are the shift equations according to components. Hence, we define the subspaces of outgoing and incoming waves $\boldsymbol{L}^{2+}_\leftarrow$ and $\boldsymbol{L}^{2-}_\leftarrow$ such that $\boldsymbol{L}^2_\leftarrow=\boldsymbol{L}^{2+}_\leftarrow\oplus\boldsymbol{L}^{2-}_\leftarrow$,

$$\boldsymbol{L}^{2+}_\leftarrow:=\{\Psi\in\boldsymbol{L}^2_\leftarrow;\Psi_2=\Psi_3=0\},\qquad\boldsymbol{L}^{2-}_\leftarrow:=\{\Psi\in\boldsymbol{L}^2_\leftarrow;\Psi_1=\Psi_4=0\},\tag{25}$$

and

$$\boldsymbol{L}^2_{\Lambda,\rightarrow}=\boldsymbol{L}^{2+}_{\Lambda,\rightarrow}\oplus\boldsymbol{L}^{2-}_{\Lambda,\rightarrow},\qquad\boldsymbol{L}^{2+}_{\Lambda,\rightarrow}:=\boldsymbol{L}^{2+}_\leftarrow,\qquad\boldsymbol{L}^{2-}_{\Lambda,\rightarrow}:=\boldsymbol{L}^{2-}_\leftarrow.\tag{26}$$

Hence, we define the wave operators $\boldsymbol{W}^\pm_\leftarrow$ at the black-hole horizon for $\Lambda\geqslant0$ and $\boldsymbol{W}^\pm_{\Lambda,\rightarrow}$ at the cosmological horizon when $\Lambda>0$, by

$$\boldsymbol{W}^\pm_\leftarrow\Psi^\pm=\lim_{t\to\pm\infty}\boldsymbol{U}(-t)\mathcal{J}_\leftarrow\mathrm{e}^{\mathrm{i}t\boldsymbol{D}_\leftarrow}\Psi^\pm\quad\text{in}\quad\boldsymbol{L}^2_{\mathrm{BH}},\qquad\Psi^\pm\in\boldsymbol{L}^{2\pm}_\leftarrow,\quad\Lambda\geqslant0\tag{27}$$

$$\boldsymbol{W}^\pm_{\Lambda,\rightarrow}\Psi^\mp=\lim_{t\to\pm\infty}\boldsymbol{U}(-t)\mathcal{J}_{\Lambda,\rightarrow}\mathrm{e}^{\mathrm{i}t\boldsymbol{D}_{\Lambda,\rightarrow}}\Psi^\mp\quad\text{in}\quad\boldsymbol{L}^2_{\mathrm{BH}},\qquad\Psi^\mp\in\boldsymbol{L}^{2\mp}_{\Lambda,\rightarrow},\quad\Lambda>0.\tag{28}$$

where $\mathcal{J}_\leftarrow$ and $\mathcal{J}_{\Lambda,\rightarrow}$ are respectively the identifying operator between $\boldsymbol{L}^2_\leftarrow$ and $\boldsymbol{L}^2_{\mathrm{BH}}$ and that between $\boldsymbol{L}^2_{\Lambda,\rightarrow}$ and $\boldsymbol{L}^2_{\mathrm{BH}}$:

$$\begin{aligned}&\mathcal{J}_\leftarrow:\Psi^\pm(r_*,\omega)\mapsto\chi_\leftarrow(r_*)r^{-1}F^{-1/4}(r)\Psi^\pm(r_*,\omega),\qquad\Psi^\pm\in\boldsymbol{L}^{2\pm}_\leftarrow,\quad\Lambda\geqslant0\\ &\mathcal{J}_{\Lambda,\rightarrow}:\Psi^\pm(r_*,\omega)\mapsto\chi_\rightarrow(r_*)r^{-1}F^{-1/4}(r)\Psi^\pm(r_*,\omega),\qquad\Psi^\pm\in\boldsymbol{L}^{2\pm}_{\Lambda,\rightarrow},\quad\Lambda>0.\end{aligned}$$

The space–time is asymptotically flat at infinity when $\Lambda = 0$. Therefore, we compare the solutions of (9) on $\boldsymbol{L}^2_{\mathrm{BH}}$ with the solution $\Psi_{\rightarrow}$ of the Dirac equation on Minkowski space–time with spherical coordinates $(\rho, \omega) \in \mathbb{R}^+_* \times [0, \pi] \times [0, 2\pi[$, putting $r_* = \rho > 0$ to avoid artificial long-range interactions:

$$\partial_t \Psi_{\rightarrow} = \mathrm{i}\boldsymbol{D}_{0,\rightarrow}\Psi_{\rightarrow} \tag{29}$$

where

$$\boldsymbol{D}_{0,\rightarrow} := \Gamma^1\left(\partial_\rho + \frac{1}{\rho}\right) + \frac{\Gamma^2}{\rho}\left(\partial_\theta + \frac{1}{2}\cot\theta\right) + \frac{\Gamma^3}{\rho\sin\theta}\partial_\varphi + \Gamma^4,$$

is self-adjoint on

$$\boldsymbol{L}^2_{0,\rightarrow} := L^2\big(\mathbb{R}^+_\rho \times S^2_\omega; \rho^2\,\mathrm{d}\rho\,\mathrm{d}\omega\big)^4$$

with the dense domain

$$\mathcal{D}(\boldsymbol{D}_{0,\rightarrow}) = H^1\big(\mathbb{R}^+_\rho \times S^2_\omega; \rho^2\,\mathrm{d}\rho\,\mathrm{d}\omega\big)^4.$$

Since the comparison of the solution of (9) on $\boldsymbol{L}^2_{\mathrm{BH}}$ with the solution of (29) on $\boldsymbol{L}^2_{0,\rightarrow}$ involves matrix-valued long-range perturbations, it is necessary to modify the free dynamic $\mathrm{e}^{\mathrm{i}t\boldsymbol{D}_{0,\rightarrow}}$ as in our previous works [13, 14]. Given $\boldsymbol{U}_{0,\rightarrow}(t)$ the Dollard-modified propagator, then we define for all $\Psi \in \boldsymbol{L}^2_{0,\rightarrow}$ the wave operator $\boldsymbol{W}^{\pm}_{0,\rightarrow}$ at infinity:

$$\boldsymbol{W}^{\pm}_{0,\rightarrow}\Psi = \lim_{t\to\pm\infty} \boldsymbol{U}(-t)\mathcal{J}_{0,\rightarrow}\boldsymbol{U}_{0,\rightarrow}(t)\Psi \quad \text{in} \quad \boldsymbol{L}^2_{\mathrm{BH}}, \tag{30}$$

where $\mathcal{J}_{0,\rightarrow}$ is the bounded identifying operator between $\boldsymbol{L}^2_{0,\rightarrow}$ and $\boldsymbol{L}^2_{\mathrm{BH}}$:

$$(\mathcal{J}_{0,\rightarrow}\Psi)(r_*, \omega) := \begin{cases} \chi_{\rightarrow}(r_*) r^{-1} F^{-1/4}(r) r_* \Psi(r_*, \omega) & r_* > 0 \\ 0 & r_* \leqslant 0, \end{cases} \qquad \forall \Psi \in \boldsymbol{L}^2_{0,\rightarrow}$$

Finally, according to [13–15], we state the theorem:

Theorem 2.1. *The wave operators* $\boldsymbol{W}^{\pm}_{\leftarrow}$, $\boldsymbol{W}^{\pm}_{\Lambda,\rightarrow}$ *and* $\boldsymbol{W}^{\pm}_{0,\rightarrow}$, *respectively defined on* $\boldsymbol{L}^{2\pm}_{\leftarrow}$, $\boldsymbol{L}^{2\mp}_{\Lambda,\rightarrow}$ *and* $\boldsymbol{L}^2_{0,\rightarrow}$ *exist and are independent of the cut-off functions* $\chi_{\leftarrow}$, $\chi_{\rightarrow}$ *and* $\chi_{\rightarrow}$ *satisfying (79). Moreover*

$$\mathrm{Ran}\big(\boldsymbol{W}^{\pm}_{\leftarrow} \oplus \boldsymbol{W}^{\pm}_{\Lambda,\rightarrow}\big) = \boldsymbol{L}^2_{\mathrm{BH}}, \qquad (\Lambda \geqslant 0)$$

and

$$\begin{aligned} &\forall \Psi^{\pm} \in \boldsymbol{L}^{2\pm}_{\leftarrow}, && \Lambda \geqslant 0, && m \geqslant 0, && \big\|\boldsymbol{W}^{\pm}_{\leftarrow}\Psi^{\pm}\big\| = \|\Psi^{\pm}\|_{\boldsymbol{L}^2_{\leftarrow}} \\ &\forall \Psi^{\mp} \in \boldsymbol{L}^{2\mp}_{\Lambda,\rightarrow}, && \Lambda > 0, && m \geqslant 0, && \big\|\boldsymbol{W}^{\pm}_{\Lambda,\rightarrow}\Psi^{\mp}\big\| = \|\Psi^{\mp}\|_{\boldsymbol{L}^2_{\Lambda,\rightarrow}} \\ &\forall \Psi \in \boldsymbol{L}^2_{0,\rightarrow}, && \Lambda = 0, && m > 0, && \big\|\boldsymbol{W}^{\pm}_{0,\rightarrow}\Psi\big\| = \|\Psi\|_{\boldsymbol{L}^2_{0,\rightarrow}}. \end{aligned}$$

3. Quantum fields

3.1. Construction of the Dirac quantum fields

To describe the quantum effects of the collapse, we need to introduce the framework of quantum field theory. For a general discussion on quantum field theory in curved space–time, we cite the following and non-exhaustive list of books: [3, 8, 18, 23]. This theory is usually defined on flat space–time. In Minkowski space–time, we have a natural choice for the vacuum state: the vacuum related to inertial observers. In this case, it is sufficient to construct a field operator which satisfies a given field equation on a Hilbert space corresponding to an inertial observer (we choose a particular Cauchy hypersurface of the space–time). Indeed, thanks to the Lorentz transformation, this construction is equivalent for all inertial observers. But in our case, we deal with a curved space–time and in general manifolds, hence we do not have the equivalent Lorentz transformations and any preferential choice for the vacuum. Then, we adopt the point of view introduced by Dimock in [6, 7]. In [7] and for the spin-1/2 fields, the author suggests a construction for local observables to globally hyperbolic manifolds which is independent (up to a net isomorphism) of the representation of the canonical anti-commutation (CAR), the choice of the spin structure and the Cauchy hypersurface.

Before explaining this construction, we define on a complex Hilbert space $(\mathfrak{H}, \langle .,.\rangle_{\mathfrak{H}})$ the Fermi–Dirac–Fock space describing the state with an arbitrary number of noninteracting charged fermions. Given a Dirac-type equation satisfied by the field f with Hamiltonian $\mathbb{H}$ defined on $\mathfrak{H}$:

$$\partial_t f = \mathrm{i}\mathbb{H} f. \tag{31}$$

We choose the spectral projectors P_+ and P_- such that

$$P_+ := \mathbf{1}_{]-\infty,0]}(\mathbb{H}), \qquad P_- := \mathbf{1}_{[0,+\infty[}(\mathbb{H}). \tag{32}$$

Then, we introduce the Fermi–Dirac–Fock space for $(\mathfrak{H}, \langle .,.\rangle_{\mathfrak{H}})$:

$$\mathfrak{F}(\mathfrak{H}) := \bigoplus_{n,m=0}^{+\infty} \mathfrak{F}^{(n,m)}, \qquad \mathfrak{F}^{(n,m)}(\mathfrak{H}) := \mathfrak{F}^{(n)}(\mathfrak{H}_+) \otimes \mathfrak{F}^{(m)}(\mathfrak{H}_-), \tag{33}$$

where

$$\mathfrak{F}^{(0)}(\mathfrak{H}_+) := \mathbb{C}, \qquad \mathfrak{F}^{(0)}(\mathfrak{H}_-) := \mathbb{C}, \qquad \mathfrak{F}^{(n)}(\mathfrak{H}_+) := \bigwedge_{k=1}^{n} \mathfrak{H}_+,$$

$$\mathfrak{F}^{(m)}(\mathfrak{H}_-) := \bigwedge_{k=1}^{m} \Upsilon \mathfrak{H}_- \tag{34}$$

and

$$\mathfrak{H} = \mathfrak{H}_+ \oplus \mathfrak{H}_-, \qquad \mathfrak{H}_+ := P_+\mathfrak{H}, \qquad \mathfrak{H}_- := P_-\mathfrak{H}. \tag{35}$$

Here, Υ is the charge conjugation (see [20] section 1.4.6). On $\mathfrak{F}(\mathfrak{H})$, we introduce $a(P_+f)$ and $a^*(P_+f)$, the particle annihilation and creation operators, and also $b(P_-f)$, $b^*(P_-f)$, the anti-particle annihilation, creation operators. We can find their rigorous definition in appendix A in [2] or in the book [4]. Therefore, we define the anti-linear quantized Dirac field operator $\mathbf{\Psi}$ and its linear adjoint $\mathbf{\Psi}^*$:

$$f \in \mathfrak{H} \longmapsto \mathbf{\Psi}(f) := a(P_+f) + b^*(\Upsilon P_- f) \in \mathcal{L}(\mathfrak{F}(\mathfrak{H})), \tag{36}$$

and

$$f \in \mathfrak{H} \longmapsto \mathbf{\Psi}^*(f) := a^*(P_+f) + b(\Upsilon P_- f) \in \mathcal{L}(\mathfrak{F}(\mathfrak{H})).$$

Moreover, these operators are bounded

$$\|\mathbf{\Psi}(f)\| = \|f\|, \qquad \|\mathbf{\Psi}^*(f)\| = \|f\|, \qquad f \in \mathfrak{H}$$

and thanks to the classical properties of the creation and annihilation operators, they satisfy the canonical anti-commutation relations (CAR):

$$\mathbf{\Psi}(f)\mathbf{\Psi}(g) + \mathbf{\Psi}(g)\mathbf{\Psi}(f) = 0, \qquad \mathbf{\Psi}^*(f)\mathbf{\Psi}^*(g) + \mathbf{\Psi}^*(g)\mathbf{\Psi}^*(f) = 0, \qquad f, g \in \mathfrak{H}$$

$$\mathbf{\Psi}^*(f)\mathbf{\Psi}(g) + \mathbf{\Psi}(g)\mathbf{\Psi}^*(f) = \langle f, g\rangle_{\mathfrak{H}}\mathbf{1}.$$

We consider the C^*-algebra $\mathfrak{U}(\mathfrak{H})$ generated by the field operators $\mathbf{\Psi}^*(f)\mathbf{\Psi}(g)$, with $f, g \in \mathfrak{H}$ and introduce the KMS state $\omega_{\mathrm{KMS}}^{\delta,\sigma}$ such that for $f, g \in \mathfrak{H}$:

$$\omega_{\mathrm{KMS}}^{\delta,\sigma}(\mathbf{\Psi}^*(f)\mathbf{\Psi}(g)) := \left\langle \mathcal{K}_{\mu,\sigma}^{\mathrm{ms}}(\mathbb{H}) f, g\right\rangle_{\mathfrak{H}}, \tag{37}$$

with, for all $x \in \mathbb{R}$

$$\mathcal{K}^{\mathrm{ms}}_{\mu,\sigma}(x) := \mu\,\mathrm{e}^{\sigma x}(1+\mu\,\mathrm{e}^{\sigma x})^{-1}, \qquad \mu := \mathrm{e}^{\sigma\delta}, \qquad \sigma > 0, \qquad \delta \in \mathbb{R}. \tag{38}$$

On the sub-algebra $\mathfrak{U}(\mathfrak{H}_+)$ (resp. $\mathfrak{U}(\mathfrak{H}_-)$) of $\mathfrak{U}(\mathfrak{H})$, the state $\omega^{\delta,\sigma}_{\mathrm{KMS}}$ provides a description of a thermodynamical equilibrium state for a gas of noninteracting Fermi particles (resp. anti-particles) with temperature $1/\sigma > 0$, chemical potential δ (resp. $-\delta$) and activity μ (resp. $1/\mu$). For more details about the physical significance of this state, the reader can read [4] and more particularly [19].

Now, according to the work of Dimock [7], we construct the algebra of local observables on a given globally hyperbolic curved space–time $\mathcal{M}$ with a foliation by a family of Cauchy hypersurfaces Π_t:

$$\mathcal{M} = \bigcup_{t\in\mathbb{R}} \Pi_t.$$

We consider a fixed hypersurface Π_t and put $\mathfrak{H} = L(\Pi_t)^4$. Using the previous definition of Dirac quantum field (36), we define on $L(\Pi_t)^4$ the quantized Dirac field $\boldsymbol{\Psi}_a$ and $\mathfrak{U}(L(\Pi_t)^2)$ the C^*-algebra generated by $\boldsymbol{\Psi}^*_a(\Phi_1)\boldsymbol{\Psi}_a(\Phi_2)$, with $\Phi_1, \Phi_2 \in L(\Pi_t)^4$. Moreover we introduce the following operator:

$$S_A : \Phi \in C_0^\infty(\mathcal{M})^4 \longmapsto S_A\Phi := \int_{\mathbb{R}} P(t,s)\Phi(s)\,\mathrm{d}s \in L(\Pi_t)^4, \tag{39}$$

where $P(t,s)$ is the isometric propagator from $L(\Pi_s)^4$ onto $L(\Pi_t)^4$, related to the Dirac field in $\mathcal{M}_{\mathrm{coll}}$. Then, we define the local quantum field in $\mathcal{M}$ by the operator:

$$\boldsymbol{\Psi}_A : \Phi \in C_0^\infty(\mathcal{M})^4 \longmapsto \boldsymbol{\Psi}_A(\Phi) := \boldsymbol{\Psi}_a(S_A\Phi), \tag{40}$$

and, for any open set $\mathcal{O} \subset \mathcal{M}$, we introduce $\mathfrak{U}(\mathcal{O})$ the C^*-algebra generated by $\boldsymbol{\Psi}_A(\Phi_1)\boldsymbol{\Psi}_A(\Phi_2)$, $\mathrm{supp}(\Phi_j) \subset \mathcal{O}$, $j = 1, 2$. Finally, we have

$$\mathfrak{U}(\mathcal{M}) = \mathrm{adh}\left(\bigcup_{\mathcal{O}} \mathfrak{U}(\mathcal{O})\right).$$

Hence according to Dimock [7], this construction is independent of the representation of the CAR, the choice of the spin structure contained in $P(t,s)$ and the fixed Cauchy hypersurface Π_t with $t \in \mathbb{R}$.

Now, we apply this procedure to the space–time outside the collapsing star $\mathcal{M}_{\mathrm{coll}}$ but also to the space–time near the future black-hole $\mathcal{M}_{\mathrm{bh}}$ and at infinity ($r_* \to +\infty$) $\mathcal{M}_{\mathrm{flat}}$ or $\mathcal{M}_{\mathrm{bh}}$, with the intention of interpreting the Hawking effect with the help of wave operators (27), (28) and (30).

For the stationary space–time $\mathcal{M}_{\mathrm{coll}}$, we have the following foliation:

$$\mathcal{M}_{\mathrm{coll}} = \bigcup_{t\in\mathbb{R}} \Pi_t, \qquad \Pi_t := \{t\} \times]z(t), +\infty[_{r_*} \times S^2_\omega.$$

We consider Π_0 and put

$$\mathfrak{H} := L^2\big(]z(0), +\infty[\times S^2_\omega, r^2F^{1/2}(r)\,\mathrm{d}r_*\,\mathrm{d}\omega\big)^4 = \boldsymbol{L}^2_0, \qquad \mathbb{H} := \boldsymbol{D}_0. \tag{41}$$

Using the previous construction, we define on $\boldsymbol{L}^2_0$ the quantized Dirac field $\boldsymbol{\Psi}_0 = \boldsymbol{\Psi}_a$ and $\mathfrak{U}(\mathfrak{H})$, the C^*-algebra generated by $\boldsymbol{\Psi}^*_0(\Phi_1)\boldsymbol{\Psi}_0(\Phi_2)$, with $\Phi_1, \Phi_2 \in \mathfrak{H}$. According to (39), we introduce $S_{\mathrm{coll}} = S_A$ with $P(0,t) = \boldsymbol{U}(0,t)$ the propagator defined in proposition 2.1. Then, we define the local quantum field in $\mathcal{M}_{\mathrm{coll}}$ by the operator

$$\boldsymbol{\Psi}_{\mathrm{coll}} : \Phi \in C_0^\infty(\mathcal{M}_{\mathrm{coll}})^4 \longmapsto \boldsymbol{\Psi}_{\mathrm{coll}}(\Phi) := \boldsymbol{\Psi}_0(S_{\mathrm{coll}}\Phi) \tag{42}$$

and also $\mathfrak{U}(\mathcal{M}_{\text{coll}})$ the closed union for all open sets $\mathcal{O} \subset \mathcal{M}_{\text{coll}}$ of $\mathfrak{U}(\mathcal{O})$ the C^*-algebra generated by $\boldsymbol{\Psi}^*_{\text{coll}}(\Phi_1)\boldsymbol{\Psi}_{\text{coll}}(\Phi_2)$, $\text{supp}(\Phi_j) \subset \mathcal{O}$, $j = 1, 2$. Then, according to (37) and (41), we define on $\mathfrak{U}(\mathcal{M}_{\text{coll}})$ a quantum state $\omega_{\mathcal{M}_{\text{coll}}}$ as following:

$$\omega_{\mathcal{M}_{\text{coll}}}(\boldsymbol{\Psi}^*_{\text{coll}}(\Phi_1)\boldsymbol{\Psi}_{\text{coll}}(\Phi_2)) := \omega^{\delta_0,\sigma_0}_{\text{KMS}}(\boldsymbol{\Psi}^*_0(S_{\text{coll}}\Phi_1)\boldsymbol{\Psi}_0(S_{\text{coll}}\Phi_2)) \tag{43}$$

$$= \left\langle \mathcal{K}^{\text{ms}}_{\mu_0,\sigma_0}(\boldsymbol{D}_0)S_{\text{coll}}\Phi_1, S_{\text{coll}}\Phi_2\right\rangle_{\mathfrak{H}}, \qquad \Phi_1, \Phi_2 \in \mathfrak{H} \tag{44}$$

with

$$\mu_0 := \mathrm{e}^{\sigma_0\delta_0}, \qquad \delta_0 \in \mathbb{R}, \qquad \sigma_0 > 0. \tag{45}$$

Indeed, we suppose that our star which is stationary in the past collapses in a gas of fermions and anti-fermions with temperature $\sigma_0^{-1} > 0$.

We describe the quantum field at the horizon of the future black hole. We consider the stationary space–time $\mathcal{M}_{\text{bh}}$ with the following foliation:

$$\mathcal{M}_{\text{bh}} = \bigcup_{t\in\mathbb{R}} \Pi_t, \qquad \Pi_t := \{t\} \times \mathbb{R}_{r_*} \times S^2_\omega.$$

By using the same procedure as above, we construct $\mathfrak{U}_{\leftarrow}(\mathcal{M}_{\text{bh}})$, the closed union for all open sets $\mathcal{O} \subset \mathcal{M}_{\text{bh}}$ of $\mathfrak{U}(\mathcal{O})$ the C^*-algebra generated by $\boldsymbol{\Psi}_{\leftarrow}(\Psi_1)\boldsymbol{\Psi}^*_{\leftarrow}(\Psi_2)$, $\Phi_1, \Phi_2 \in \boldsymbol{L}^2_{\leftarrow}$ where

$$\boldsymbol{\Psi}_{\leftarrow} : \Phi \in C^\infty_0(\mathcal{M}_{\text{bh}})^4 \longmapsto \boldsymbol{\Psi}_{\leftarrow}(\Phi) := \boldsymbol{\Psi}_{-}(S_{\leftarrow}\Phi), \tag{46}$$

and

$$S_{\leftarrow} := S_A, \qquad P(0,t) := \mathrm{e}^{-\mathrm{i}t\boldsymbol{D}_{\leftarrow}}. \tag{47}$$

Here $\boldsymbol{\Psi}_{-}(\Phi)$ with $\Phi \in \boldsymbol{L}^2_{\leftarrow}$ is the quantum Dirac field defined on the hypersurface $\mathbb{R}_{r_*} \times S^2_\omega$. By using (37), we consider the Hawking thermal state:

$$\omega^{\delta,\sigma}_{\text{Haw}}(\boldsymbol{\Psi}^*_{\leftarrow}(\Phi_1)\boldsymbol{\Psi}_{\leftarrow}(\Phi_2)) := \omega^{\delta,\sigma}_{\text{KMS}}(\boldsymbol{\Psi}^*_{-}(S_{\leftarrow}\Phi_1)\boldsymbol{\Psi}_{-}(S_{\leftarrow}\Phi_2)), \quad \Phi_1, \Phi_2 \in C^\infty_0(\mathcal{M}_{\text{bh}})^4 \tag{48}$$

$$= \left\langle \mathcal{K}^{\text{ms}}_{\mu,\sigma}(\boldsymbol{D}_{\leftarrow})S_{\leftarrow}\Phi_1, S_{\leftarrow}\Phi_2\right\rangle_{\boldsymbol{L}^2_{\leftarrow}}, \tag{49}$$

with

$$\mu := \mathrm{e}^{\sigma\delta}, \quad \delta \in \mathbb{R}, \quad \sigma > 0. \tag{50}$$

Finally, we introduce the quantum fields at infinity when $r_* \to +\infty$. According to Λ which is respectively positive or zero (cosmological horizon or asymptotically flat space–time), we consider the stationary space–time

$$\mathcal{M}_{\text{bh}} = \mathbb{R}_t \times \mathbb{R}_{r_*} \times S^2_\omega, \qquad \mathcal{M}_{\text{flat}} := \mathbb{R}_t \times \mathbb{R}^+_{r_*} \times S^2_\omega.$$

As above, using the Fermi–Dirac–Fock quantization on $\mathbb{R}_{r_*} \times S^2_\omega$ or $\mathbb{R}^+_{r_*} \times S^2_\omega$, we define the fields $\boldsymbol{\Psi}_{\Lambda,+}(\Phi_1)$ with $\Phi_1 \in \boldsymbol{L}^2_{\Lambda,\rightarrow}$ or $\boldsymbol{\Psi}_{0,+}(\Phi_1)$ with $\Phi_1 \in \boldsymbol{L}^2_{0,\rightarrow}$. Hence, we construct $\mathfrak{U}_{\rightarrow}(\mathcal{M}_{\text{bh}})$ and $\mathfrak{U}_{\rightarrow}(\mathcal{M}_{\text{flat}})$. The algebra $\mathfrak{U}_{\rightarrow}(\mathcal{M}_{\text{bh}})$ is the closed union for all open sets $\mathcal{O} \subset \mathcal{M}_{\text{bh}}$ of the C^*-algebras $\mathfrak{U}_{\rightarrow}(\mathcal{O})$ generated by $\boldsymbol{\Psi}^*_{\Lambda,\rightarrow}(\Phi_1)\boldsymbol{\Psi}_{\Lambda,\rightarrow}(\Phi_1)$ with $\Phi_1,\ \Phi_2 \in \boldsymbol{L}^2_{\Lambda,\rightarrow}$

$$\boldsymbol{\Psi}_{\Lambda,\rightarrow} : \Phi \in C^\infty_0(\mathcal{M}_{\text{bh}})^4 \longmapsto \boldsymbol{\Psi}_{\Lambda,\rightarrow}(\Phi) := \boldsymbol{\Psi}_{\Lambda,+}(S_{\Lambda,\rightarrow}\Phi), \qquad \Lambda > 0 \tag{51}$$

and

$$S_{\Lambda,\rightarrow} := S_A, \qquad P(0,t) := \mathrm{e}^{-\mathrm{i}t\boldsymbol{D}_{\Lambda,\rightarrow}}, \qquad \Lambda > 0. \tag{52}$$

As to the algebra $\mathfrak{U}_{\rightarrow}(\mathcal{M}_{\text{flat}})$, it is the closed union for all $\mathcal{O} \subset \mathcal{M}_{\text{flat}}$ of the C^*-algebras $\mathfrak{U}_{\rightarrow}(\mathcal{O})$ generated by $\boldsymbol{\Psi}^*_{0,\rightarrow}(\Phi_1)\boldsymbol{\Psi}_{0,\rightarrow}(\Phi_1)$ with $\Phi_1, \Phi_2 \in \boldsymbol{L}^2_{0,\rightarrow}$,

$$\boldsymbol{\Psi}_{0,\rightarrow} : \Phi \in C^\infty_0(\mathcal{M}_{\text{flat}})^4 \longmapsto \boldsymbol{\Psi}_{0,\rightarrow}(\Phi) := \boldsymbol{\Psi}_{0,+}(S_{0,\rightarrow}\Phi) \tag{53}$$

and

$$S_{0,\rightarrow} := S_A, \qquad P(0,t) := \boldsymbol{U}_{0,\rightarrow}(-t), \tag{54}$$

where $\boldsymbol{U}_{0,\rightarrow}$ is the Dollard-modified propagator. With (37), the thermal states on each algebra $\mathfrak{U}_{\rightarrow}(\mathcal{M}_{\text{bh}})$ and $\mathfrak{U}_{\rightarrow}(\mathcal{M}_{\text{flat}})$ are given by

$$\forall \Phi_1, \Phi_2 \in C_0^\infty(\mathcal{M}_{\text{bh}}),$$
$$\omega_{\text{KMS}}^{\delta_0,\sigma_0}(\boldsymbol{\Psi}^*_{\Lambda,\rightarrow}(\Phi_1)\boldsymbol{\Psi}_{\Lambda,\rightarrow}(\Phi_1)) = \left\langle \mathcal{K}^{\text{ms}}_{\mu_0,\sigma_0}(\boldsymbol{D}_{\Lambda,\rightarrow})S_{\Lambda,\rightarrow}\Phi_1, S_{\Lambda,\rightarrow}\Phi_2\right\rangle_{\boldsymbol{L}^2_{\Lambda,\rightarrow}},$$

with $\Lambda > 0$, and

$$\forall \Phi_1, \Phi_2 \in C_0^\infty(\mathcal{M}_{\text{flat}}),$$
$$\omega_{\text{KMS}}^{\delta_0,\sigma_0}(\boldsymbol{\Psi}^*_{0,\rightarrow}(\Phi_1)\boldsymbol{\Psi}_{0,\rightarrow}(\Phi_1)) = \left\langle \mathcal{K}^{\text{ms}}_{\mu_0,\sigma_0}(\boldsymbol{D}_{0,\rightarrow})S_{0,\rightarrow}\Phi_1, S_{0,\rightarrow}\Phi_2\right\rangle_{\boldsymbol{L}^2_{0,\rightarrow}}.$$

3.2. Hawking effect

The state

$$\omega_{\mathcal{M}_{\text{coll}}}(\boldsymbol{\Psi}^*_{\text{coll}}(\Phi_1)\boldsymbol{\Psi}_{\text{coll}}(\Phi_2)), \qquad \Phi_j \in C_0^\infty(\mathcal{M}_{\text{coll}})^4, \quad j = 1, 2,$$

gives the information about the quantum fluctuations in a region of $\mathcal{M}_{\text{coll}}$. But, we are interested in the investigation of this previous state at the last moment of gravitational collapse when the detector is fixed with respect to the variables (r_*, ω). As this collapsing star becomes a black hole, the detector at rest receives the information from the creation of the black hole when this proper time $t = \infty$. Hence, we put

$$\Phi_j^T(t, r_*, \omega) := \Phi_j(t - T, r_*, \omega), \qquad \Phi_j \in C_0^\infty(\mathcal{M}_{\text{coll}})^4, \qquad j = 1, 2,$$

and state the main theorem about the behaviour of $\omega_{\mathcal{M}_{\text{coll}}}$ at the last moment of the collapse:

Theorem 3.1. *Given $\Phi_j \in C_0^\infty(\mathcal{M}_{\text{coll}})^4$, $j = 1, 2$, then we have for $\Lambda \geqslant 0$,*

$$\lim_{T\to+\infty} \omega_{\mathcal{M}_{\text{coll}}}\big(\boldsymbol{\Psi}^*_{\text{coll}}\big(\Phi_1^T\big)\boldsymbol{\Psi}_{\text{coll}}\big(\Phi_2^T\big)\big) = \omega^{\delta,\sigma}_{\text{Haw}}(\boldsymbol{\Psi}^*_{\leftarrow}(\boldsymbol{\Omega}^-_{\leftarrow}\Phi_1)\boldsymbol{\Psi}_{\leftarrow}(\boldsymbol{\Omega}^-_{\leftarrow}\Phi_2))$$
$$+\, \omega_{\text{KMS}}^{\delta_0,\sigma_0}(\boldsymbol{\Psi}^*_{\Lambda,\rightarrow}(\boldsymbol{\Omega}^-_{\Lambda,\rightarrow}\Phi_1)\boldsymbol{\Psi}_{\Lambda,\rightarrow}(\boldsymbol{\Omega}^-_{\Lambda,\rightarrow}\Phi_2)),$$

with

$$T_{\text{Haw}} = \frac{1}{\sigma} = \frac{2\pi}{\kappa_0}, \qquad \delta = \frac{qQ}{r_0}.$$

Let us interpret the previous theorem.We know that the state $\omega_{\mathcal{M}_{\text{coll}}}$ represents the response of a detector at rest in Schwarzschild variables at time T. This detector is initially put in a state that corresponds for a static observer to a gas of fermions, where the particles do not interact between themselves and are defined by the constants of temperature $\sigma_0 > 0$ and chemical potential δ_0.

As $T = +\infty$, the detector measures the quantum fluctuations related to $\omega_{\mathcal{M}_{\text{coll}}}$ when the star becomes a black hole. In this situation, the detector measures two types of information: about fields coming from the past infinity (and falling into the black hole) and about fields coming from the future horizon of the black hole (going to the future infinity).

Since the state $\omega_{\text{KMS}}^{\delta_0,\sigma_0}$ contains the wave operators $\boldsymbol{\Omega}^-_{\Lambda,\rightarrow}$ in its definition, $\omega_{\text{KMS}}^{\delta_0,\sigma_0}$ gives information about fields of the first type. It means that the detector measures a quantum fluctuation coming from the past infinity which is interpreted by a static observer as a flux of particles with the same characteristics as the initial state.

In the same way, since $\omega_{\text{Haw}}^{\delta,\sigma}$ contains the wave operators $\boldsymbol{\Omega}_{\leftarrow}^{-}$ in its definition, this state gives information about the fields coming from the future black-hole horizon. Indeed, the detector measures the emergence of the thermal state with temperature

$$T_{\text{Haw}} = \frac{1}{\sigma} = \frac{2\pi}{\kappa_0}$$

which is interpreted by a static observer as a flux of particles and anti-particles with charge density

$$\rho_{\text{Haw}} := \frac{1}{\pi} q\delta = \frac{q^2 Q}{\pi r_0}.$$

As in our previous study [15], the black hole will preferentially emit particles whose charge is of the same sign as its own charge.

We remark that the result is independent of the story of the collapse, the boundary condition (the characteristic of the star surface) and also the initial state since we proved the same result in [15] by supposing that the initial state is Boulware type in the past. This is a *no hair* result.

Moreover, the previous theorem is valid when $\Lambda \geqslant 0$. When $\Lambda > 0$, we consider the de Sitter–Reissner–Nordstrøm space–time outside the star before and during the collapse. Let us recall that this curved space–time has a cosmological horizon at infinity. In this case, Gibbons and Hawking have proved in [10] that an observer following any timelike geodesic measures an isotropic background of thermal radiation coming from the past cosmological horizon with the (Gibbons–Hawking) temperature

$$T_{\text{GH}} = \frac{2\pi}{\kappa_+}.$$

Here κ_+ is the surface gravity at the cosmological horizon defined in (2). Hence, a static observer interprets this radiation as a flux of particles coming from the past cosmological horizon with temperature $T_{\text{GH}} = \sigma_{\text{GH}}^{-1}$ and chemical potential δ_{GH}. Hence, we define the quantum state $\omega_{\mathcal{M}_{\text{coll}}}$ outside the collapsing star. On $\mathfrak{U}(\mathcal{M}_{\text{coll}})$ and for all $\Phi_1, \Phi_2 \in \boldsymbol{L}_0^2$ we have

$$\begin{aligned}
\omega_{\mathcal{M}_{\text{coll}}}(\boldsymbol{\Psi}_{\text{coll}}^*(\Phi_1)\boldsymbol{\Psi}_{\text{coll}}(\Phi_2)) &:= \omega_{\text{KMS}}^{\delta_0,\sigma_0}(\boldsymbol{\Psi}_0^*(S_{\text{coll}}\Phi_1)\boldsymbol{\Psi}_0(S_{\text{coll}}\Phi_2)) \\
&= \left\langle \mathcal{K}_{\mu_0,\sigma_0}^{\text{ms}}(\boldsymbol{D}_0) S_{\text{coll}}\Phi_1, S_{\text{coll}}\Phi_2 \right\rangle_{\boldsymbol{L}_0^2}, \\
&= \left\langle \boldsymbol{W}_{\Lambda,D}^{-} \mathcal{K}_{\mu_0,\sigma_0}^{\text{ms}}(\boldsymbol{D}_0) S_{\text{coll}}\Phi_1, \boldsymbol{W}_{\Lambda,D}^{-} S_{\text{coll}}\Phi_2 \right\rangle_{\boldsymbol{L}_{\Lambda,\rightarrow}^2}, \\
&= \left\langle \mathcal{K}_{\mu_0,\sigma_0}^{\text{ms}}(\boldsymbol{D}_{\Lambda,\rightarrow}) S_{\Lambda,\rightarrow} \boldsymbol{W}_{\Lambda,D}^{-}\Phi_1, S_{\Lambda,\rightarrow}\boldsymbol{W}_{\Lambda,D}^{-}\Phi_2 \right\rangle_{\boldsymbol{L}_{\Lambda,\rightarrow}^2} \\
&= \omega_{\text{KMS}}^{\delta_0,\sigma_0}(\boldsymbol{\Psi}_{\Lambda,\rightarrow}^*(\boldsymbol{W}_{\Lambda,D}^{-}\Phi_1)\boldsymbol{\Psi}_{\Lambda,\rightarrow}(\boldsymbol{W}_{\Lambda,D}^{-}\Phi_2)),
\end{aligned}$$

where $\boldsymbol{W}_{\Lambda,D}^{-}$ is the wave operator linking the dynamic outside the star before the beginning of the collapse and the free dynamic at the past cosmological horizon (see (80), (142) and (143) for the definition). Hence, in the case of a cosmological model with a positive cosmological constant, the only physically relevant choice for the σ_0 and δ_0 is

$$\sigma_0 = \sigma_{\text{GH}} = T_{\text{GH}}^{-1} = \frac{\kappa_+}{2\pi}, \qquad \delta_0 = \delta_{\text{GH}}.$$

4. Proof of theorem 3.1

This section is devoted to the proof of theorem 2. In order to demonstrate this theorem (section 4.3), we prove the following sharp result:

Theorem 4.1. *Given* $f \in \boldsymbol{L}^2_{\mathrm{BH}}$, *if* $\Lambda \geqslant 0$, *then*

$$\lim_{T\to+\infty}\left\langle \mathcal{K}^{\mathrm{ms}}_{\mu_0,\sigma_0}(\boldsymbol{D}_0)\boldsymbol{U}(0,T)f, \boldsymbol{U}(0,T)f\right\rangle_{\mathfrak{H}} = \left\langle \mathcal{K}^{\mathrm{ms}}_{\mu_0,\sigma_0}(\boldsymbol{D}_{\Lambda,\rightarrow})\boldsymbol{\Omega}^-_{\Lambda,\rightarrow} f, \boldsymbol{\Omega}^-_{\Lambda,\rightarrow} f\right\rangle_{\boldsymbol{L}^2_{\Lambda,\rightarrow}} + \left\langle \mathcal{K}^{\mathrm{ms}}_{\mu,\sigma}(\boldsymbol{D}_{\leftarrow})\boldsymbol{\Omega}^-_{\leftarrow} f, \boldsymbol{\Omega}^-_{\leftarrow} f\right\rangle_{\boldsymbol{L}^2_{\leftarrow}} \quad (55)$$

with $\mu = \mathrm{e}^{\sigma\delta}$, $\delta := \frac{qQ}{r_0}$, $\sigma = \frac{2\pi}{\kappa_0}$, $\boldsymbol{\Omega}^-_{\leftarrow} := (\boldsymbol{W}^-_{\leftarrow})^*$, $\boldsymbol{\Omega}^-_{\Lambda,\rightarrow} := (\boldsymbol{W}^-_{\Lambda,\rightarrow})^*$, $\boldsymbol{\Omega}^-_{0,\rightarrow} := (\boldsymbol{W}^-_{0,\rightarrow})^*$, *where* $\boldsymbol{W}^-_{\leftarrow}$, $\boldsymbol{W}^-_{\Lambda,\rightarrow}$, $\boldsymbol{W}^-_{0,\rightarrow}$ *are the wave operators respectively defined in (27), (28) and (30).*

To prove the limit (55), we use the spherical symmetry property of the geometrical framework. Indeed, we introduce the spin-weighted harmonics to reduce our study to a family of one dimensional problems. This is the purpose of the following section.

4.1. Reduction to a simpler problem thanks to the spherical symmetry

Given $Y^l_{\pm\frac{1}{2},n}$ the spin-weighted harmonics (see [9, 13]) such that the families

$$\left\{Y^l_{\frac{1}{2},n};\,(l,n)\in\mathcal{I}\right\},\qquad \left\{Y^l_{-\frac{1}{2},n};\,(l,n)\in\mathcal{I}\right\},\qquad \mathcal{I} := \left\{(l,n) : l-\tfrac{1}{2}\in\mathbb{N}, l-|n|\in\mathbb{N}\right\},$$

form a Hilbert basis of $L^2\left(S^2_\omega\right)$, and each Y^l_{sn}, $s = \pm 1/2$ satisfies the recurrence relations,

$$\partial_\theta Y^l_{sn}(\omega) \mp \frac{n - s\cos\theta}{\sin\theta} Y^l_{sn}(\omega) = \begin{vmatrix} -\mathrm{i}\sqrt{(l\pm s)(l\mp s+1)}\,Y^l_{s\mp 1,n}(\omega), & \pm l > -s. \\ 0,\ l = \mp s. & \end{vmatrix}, \quad (56)$$

$$\partial_\varphi Y^l_{sn}(\omega) = -\mathrm{i}nY^l_{sn}(\omega). \quad (57)$$

Afterwards, we introduce the following Hilbert spaces:

$$\left(L^2_t := L^2(]z(t), +\infty[_{r_*}, \mathrm{d}r_*)^4, \|.\|_t\right), \qquad 0 \leqslant t \quad (58)$$

$$\left(L^2_{\mathbb{R}} := L^2(\mathbb{R}_{r_*}, \mathrm{d}r_*)^4, \|.\|\right), \quad (59)$$

$$L^2_{\mathrm{BH}} := L^2(\mathbb{R}_{r_*}, r^2F^{1/2}(r)\,\mathrm{d}r_*)^4 = \mathcal{P}_r L^2_{\mathbb{R}}, \quad (60)$$

with

$$\mathcal{P}_r : \Psi \mapsto r^{-1}F^{-1/4}\Psi. \quad (61)$$

So, we express $\boldsymbol{L}^2_t$ and $\boldsymbol{L}^2_{\mathrm{BH}}$ as a direct sum:

$$\boldsymbol{L}^2_t = \bigoplus_{(l,n)\in\mathcal{I}} \mathcal{E}^\nu_{ln} L^2_t, \qquad \boldsymbol{L}^2_{\mathrm{BH}} = \bigoplus_{(l,n)\in\mathcal{I}} \mathcal{E}^\nu_{ln} L^2_{\mathbb{R}}. \quad (62)$$

where

$$\mathcal{E}^\nu_{ln} : \Psi_{ln} \in L^2_t \mapsto \mathrm{e}^{-\mathrm{i}\frac{\nu}{2}\gamma^5}\mathcal{P}_r\Psi_{ln} \otimes_4 Y_{ln} \in \boldsymbol{L}^2_t \quad (63)$$

with

$$v \otimes_4 u := (u_1v_1, u_2v_2, u_3v_3, u_4v_4), \qquad \forall u, v \in \mathbb{C}^4,$$
$$Y_{ln} := \left(Y^l_{-\frac{1}{2},n}, Y^l_{\frac{1}{2},n}, Y^l_{-\frac{1}{2},n}, Y^l_{\frac{1}{2},n}\right). \quad (64)$$

Defining the following restriction operator $\mathcal{R}^\nu_{ln}$ such that

$$\mathcal{R}^\nu_{ln} : \Psi \in \boldsymbol{L}^2_t \mapsto \mathrm{e}^{\mathrm{i}\frac{\nu}{2}\gamma^5}\mathcal{P}_r^{-1}\Psi_{ln} \in L^2_t, \qquad \Psi_{ln} := \langle\Psi, Y_{ln}\rangle \quad (65)$$

and using (56), (57) for $s = \pm 1/2$, we obtain the following decomposition:

$$\boldsymbol{D}_t = \bigoplus_{(l,n)\in\mathcal{I}} \mathcal{E}^\nu_{ln} D_{V_{l,\nu},t} \mathcal{R}^\nu_{ln} - \frac{qQ}{r_0}, \tag{66}$$

$$D_{V_{l,\nu},t} := \Gamma^1 \partial_{r_*} + V_{l,\nu}, \qquad V_{l,\nu} = qQ\left(\frac{1}{r_0} - \frac{1}{r}\right) - \sqrt{F(r)}\left(mA_\nu + \frac{\mathrm{i}}{r}\Gamma^2(l+1/2)\right), \tag{67}$$

$$A_\nu := \begin{pmatrix} 0 & a_\nu \\ \bar{a}_\nu & 0 \end{pmatrix}, \qquad a_\nu := \mathrm{diag}(\mathrm{i}\,\mathrm{e}^{\mathrm{i}\nu}, \mathrm{i}\,\mathrm{e}^{\mathrm{i}\nu}), \qquad Z(t) = \sqrt{\frac{1-\dot{z}(t)}{1+\dot{z}(t)}}, \tag{68}$$

$$\mathcal{D}\big(D_{V_{l,\nu},t}\big) = \big\{\Psi \in L^2_t;\, D_{V_{l,\nu},t}\Psi \in L^2_t, \quad Z(t)\Psi_2(z(t)) = \Psi_4(z(t)), \\ \Psi_1(z(t)) = -Z(t)\Psi_3(z(t))\big\}. \tag{69}$$

For $\Phi \in L^2(B, \mathrm{d}r_*)^4$, $B \subset \mathbb{R}$, we define a L^2-extension such that

$$\|\Phi\|_{L^2(B,\mathrm{d}r_*)^4} = \|[\Phi]_L\|, \qquad [\Phi]_L(r_*) := \begin{cases} \Phi(r_*) & r_* \in B \\ 0 & r_* \in \mathbb{R}\backslash B \end{cases}.$$

In the same way, we introduce

$$0 \leqslant t, \qquad H^1_t := \big\{\Phi \in L^2_t,\ \partial_{r_*}\Phi \in L^2_t\big\}, \qquad H^1_{\mathbb{R}} := \big\{\Phi \in L^2_{\mathbb{R}},\ \partial_{r_*}\Phi \in L^2_{\mathbb{R}}\big\},$$

and a H^1-extension such that for $\Phi \in H^1_t$ we have

$$[\Phi]_H \in H^1_{\mathbb{R}}, \qquad [\Phi]_H(r_*) := \begin{cases} \Phi(r_*) & r_* \in]z(t), +\infty[_{r_*} \\ \Phi(2z(t) - r_*) & r_* \in \mathbb{R}\backslash]z(t), +\infty[_{r_*}. \end{cases}$$

For dynamic $D_{V_{l,\nu},t}$, we set proposition VI.2 in [2] which gives a unique solution expressed with propagator $U_{V_{l,\nu}}(t,s)$ of

$$\partial_t \Phi = \mathrm{i} D_{V_{l,\nu},t}\Phi, \qquad t \in \mathbb{R}, \qquad r_* > z(t), \tag{70}$$

$$\Phi_4(t, z(t)) = Z(t)\Phi_2(t, z(t)), \qquad \Phi_1(t, z(t)) = -Z(t)\Phi_3(t, z(t)), \tag{71}$$

$$\Phi(t = s, .) = \Phi_s(.) \in L^2_s. \tag{72}$$

Proposition 4.1. *If $\Phi_s \in \mathcal{D}(D_{V_{l,\nu},s})$, then there exists a unique solution*

$$[\Phi(.)]_H = \big[U_{V_{l,\nu}}(., s)\Phi_s\big]_H \in C^1\big(\mathbb{R}_t, L^2_{\mathbb{R}}\big) \cap C^0\big(\mathbb{R}_t, H^1_{\mathbb{R}}\big)$$

of (70), (71) and (72) with

$$\Phi(t) \in \mathcal{D}\big(D_{V_{l,\nu},t}\big).$$

Moreover,

$$\|\Phi(t)\|_t = \|\Phi_s\|_s \tag{73}$$

and $U_{V_{l,\nu}}(t,s)$ can be extended in an isometric strongly continuous propagator from L^2_s onto. L^2_t.

Operators (63) and (65) are very useful to express $\boldsymbol{U}(t,s)$ defined in proposition (2.1) with the help of $U_{V_{l,\nu}}(t,s)$:

$$\boldsymbol{U}(t,s) = \mathrm{e}^{\mathrm{i}(s-t)\frac{qQ}{r_0}} \bigoplus_{(l,n)\in\mathcal{I}} \mathcal{E}^\nu_{ln} U_{V_{l,\nu}}(t,s)\mathcal{R}^\nu_{ln} : \boldsymbol{L}^2_s = \bigoplus_{(l,n)\in\mathcal{I}} \mathcal{E}^\nu_{ln} L^2_s \to \boldsymbol{L}^2_t = \bigoplus_{(l,n)\in\mathcal{I}} \mathcal{E}^\nu_{ln} L^2_t. \tag{74}$$

Given a potential $V \in L^\infty(\mathbb{R}_{r_*})$ and an interval $B := (a, +\infty)$ or $B := (-\infty, a)$ and $V \in L^\infty(\mathbb{R}_{r_*})$, then, we define on $L^2(B)^4$ the self-adjoint operator $D_{V,B}$ with the dense domain $\mathcal{D}(D_{V,B})$ such that

$$D_{V,B} = \Gamma^1 \partial_{r_*} + V, \tag{75}$$

$$\mathcal{D}(D_{V,B}) = \{\Phi \in L^2(B)^4, \quad D_{V,B}\Phi \in L^2(B)^4, \quad r_* \in \partial B \Rightarrow \vec{n}\gamma^1\Phi(r_*) = \mathrm{i}\Phi(r_*)\}, \tag{76}$$

where Γ^1 is given by (14) and $\vec{n}$ is the outgoing normal of B. Using Kato–Rellich and spectral theorem, it is easy to find a unique solution of

$$\partial_t \Phi = \mathrm{i} D_{V,B}\Phi, \qquad \Phi(0) = \Psi_0. \tag{77}$$

using propagator $U_{V,B}(t)$:

Proposition 4.2. *Given $\Phi_0 \in \mathcal{D}(D_{V,B})$, then there exists a unique solution*

$$\Phi(.) = U_{V,B}(.)\Phi_0 \in C^0(\mathbb{R}_t, \mathcal{D}(D_{V,B})) \cap C^1(\mathbb{R}_t, L^2(B)^4)$$

and

$$\|\Phi(t)\| = \|\Phi_0\|.$$

Moreover, $U_{V,B}(t)$ can be extended, by density and continuity, to a strongly unitary group on $L^2(B)^4$.

Thus, we can express the propagator $\boldsymbol{U}(t)$ defined in proposition 2.2 with the help of $U_{V,B}(t)$ and operators (63) and (65):

$$\boldsymbol{U}(t) = \mathrm{e}^{-\mathrm{i}t\frac{qQ}{r_0}} \bigoplus_{(l,n)\in\mathcal{I}} \mathcal{E}^{\nu}_{ln} U_{V_{l,\nu},\mathbb{R}}(t) \mathcal{R}^{\nu}_{ln}. \tag{78}$$

Now, we introduce the useful wave operators for the next part. We choose a cut-off function $\chi \in C^\infty(\mathbb{R}_{r_*})$, such that

$$\exists\, a, b \in \mathbb{R}, \qquad -\infty < a < b < +\infty, \qquad \chi(r_*) = \begin{cases} 1 & r_* < a \\ 0 & r_* > b, \end{cases} \tag{79}$$

and the subspaces $L^{2+}_{\mathbb{R}}$ and $L^{2-}_{\mathbb{R}}$ of $L^2_{\mathbb{R}}$ with the following properties:

$$L^{2+}_{\mathbb{R}} = \left\{\Phi \in L^2_{\mathbb{R}}; \Phi_2 \equiv \Phi_3 \equiv 0\right\}, \qquad L^{2-}_{\mathbb{R}} = \left\{\Phi \in L^2_{\mathbb{R}}; \Phi_1 \equiv \Phi_4 \equiv 0\right\}.$$

Hence, we state

Lemma 4.1. *Given $V = V_{l,\nu}$ to simplify the notation. The wave operators*

$$\begin{aligned} W^{\pm}_{0,\mathbb{R}} &= s - \lim_{t\to\pm\infty} U_{0,\mathbb{R}}(-t)\chi U_{V,\mathbb{R}}(t), \qquad in \quad L^2_{\mathbb{R}} \\ W^{\pm}_{V,[z(0),+\infty[} &= s - \lim_{t\to\pm\infty} U_{V,[z(0),+\infty[}(-t)(1-\chi)U_{V,\mathbb{R}}(t) \qquad in \quad L^2_0 \end{aligned} \tag{80}$$

exist and are independent of χ satisfying (79). Moreover

$$\mathrm{Ran}\left(W^{\pm}_{0,\mathbb{R}}\right) = L^{2\pm}_{\mathbb{R}}, \qquad \mathrm{Ran}\left(W^{\pm}_{V,[z(0),+\infty[}\right) = P_{ac}(D_{V,[z(0),+\infty[})L^2_0 \tag{81}$$

where $P_{ac}(D_{V,[z(0),+\infty[})$ is the projector on the absolutely continuous subspace of $D_{V,[z(0),+\infty[}$.

Proof. See lemma 6.3 in [15]. □

By using operators (63) and (65), we easily remark that

$$\mathcal{P}_r(\boldsymbol{W}^{-}_{\leftarrow})^* = \bigoplus_{(l,n)\in\mathcal{I}} \mathcal{E}^{\nu}_{ln} W^{-,l}_{0,\mathbb{R}} \mathcal{R}^{\nu}_{ln}. \tag{82}$$

4.2. Proof of theorem 4.1

Firstly, we describe the main ideas of the demonstration. Our proof uses some results from some previous works: the sharp study of the backward propagator $\boldsymbol{U}(0,T)$ [15], the scattering theory in the eternal charged black hole [13–15]. With operators (63) and (65) we obtain the important relation:

$$\mathcal{K}^{\text{ms}}_{\mu_0,\sigma_0}(\boldsymbol{D}_0)\boldsymbol{U}(0,T) = \mathrm{e}^{\mathrm{i}T\delta} \bigoplus_{(l,n)\in\mathcal{I}} \mathcal{E}^{\nu}_{ln}\mathcal{K}^{\text{ms}}_{1,\sigma_0}\big(D_{V_{l,\nu},0}\big)U_{V_{l,\nu}}(0,T)\mathcal{R}^{\nu}_{ln}, \qquad \delta := \frac{qQ}{r_0}.$$

Hence, using the spherical invariance, we reduce our study to a one-dimensional problem, i.e. the study of $\mathcal{K}^{\text{ms}}_{1,\sigma_0}\big(D_{V_{l,\nu},0}\big)U_{V_{l,\nu}}(0,T)$ as $T \to +\infty$. Now, we forget subscripts ln and ν to simplify the notation. As in [15], we split our investigation into two parts, thanks to the following cut-off function $\mathcal{J} \in C^{\infty}\big(\mathbb{R}_{r_*}\big)$ satisfying

$$\exists\, a, b \in \mathbb{R}, \qquad 0 < a < b < 1 \qquad \mathcal{J}(r_*) = \begin{cases} 1 & r_* < a \\ 0 & r_* > b. \end{cases} \tag{83}$$

Henceforth, we have

$$\mathcal{K}^{\text{ms}}_{1,\sigma_0}(D_{V,0})U_V(0,T) = \mathcal{K}^{\text{ms}}_{1,\sigma_0}(D_{V,0})\mathcal{J}U_V(0,T) + \mathcal{K}^{\text{ms}}_{1,\sigma_0}(D_{V,0})(1-\mathcal{J})U_V(0,T), \tag{84}$$

where the two last terms are asymptotically orthogonal as $T \to +\infty$. Far from the star and thanks to the hyperbolicity, we have

$$\mathcal{K}^{\text{ms}}_{1,\sigma_0}(D_{V,0})(1-\mathcal{J})U_V(0,T) = \mathcal{K}^{\text{ms}}_{1,\sigma_0}(D_{V,0})(1-\mathcal{J})U_{V,\mathbb{R}}(-T),$$

where $U_{V,\mathbb{R}}$ is defined by proposition 4.2. Since this last propagator is straight linked with $\boldsymbol{U}(t)$ by formula (78), the scattering theory in the eternal charged black hole is very useful to conclude. Near the star, we prove that

$$\mathcal{K}^{\text{ms}}_{1,\sigma_0}(D_{V,0})\mathcal{J}U_V(0,T)f \sim \mathbf{1}_{[\delta,+\infty[}(D_{V,0})\mathcal{J}U_V(0,T)f, \qquad T \to +\infty, \qquad f \in L^2_{\mathbb{R}}. \tag{85}$$

This relation requires some technical lemmas, mainly of compactness. Thus, the weak convergence of $\mathcal{J}U_V(0,T)$ as $T \to +\infty$ is an important property to obtain the result. To conclude the proof, we use a result from a previous work [15]:

$$\mathbf{1}_{[\delta,+\infty[}(D_{V,0})\mathcal{J}U_V(0,T)f \sim \big\langle \mathcal{K}^{\text{ms}}_{1,\sigma}(D_{0,\mathbb{R}})W^-_{0,\mathbb{R}}f, W^-_{0,\mathbb{R}}f\big\rangle_{L^2_{\mathbb{R}}}, \qquad T \to +\infty, \qquad f \in L^2_{\mathbb{R}}, \tag{86}$$

seeing that the wave operator $W^-_{0,\mathbb{R}}$ is linked with $\boldsymbol{W}^-_{\leftarrow}$ by formula (82).

We introduce some notation:

$$D_{V,0} := D_{V,[z(0),+\infty[}\,, \qquad L^2_0 := L^2\big([z(0),+\infty[_{r_*}, \mathrm{d}r_*\big)^4. \tag{87}$$

For $g := (g_1, g_2, g_3, g_4) \in L^2_{\mathbb{R}}$,

$$g^T(.) := g(.-T), \qquad T \geqslant 0$$

and

$$G(r_*) := \frac{1}{\sqrt{-\kappa_0 r_*}}\,{}^t(-g_3, 0, 0, g_2)\left(-\frac{1}{2\kappa_0}\ln(-r_*) + \frac{1}{2\kappa_0}\ln\big(C_{\kappa_0}\big) + \frac{1}{2}\right), \qquad r_* < 0,$$

with $C_{\kappa_0} > 0$. To obtain relation (85), we set and prove some lemmas. For this, we use the notation introduced by formulae (66), (67), (75), (76) and propositions 4.1 and 4.2.

Lemma 4.2. *Given* ${}^t(0, g_2, g_3, 0) \in C_0^{\infty}(\mathbb{R})^4$*, then for* $\Lambda \geqslant 0$*:*

$$\lim_{T\to+\infty} \big\langle \big(\mathcal{K}^{\text{ms}}_{\mu_0,\sigma_0}(D_{0,\mathbb{R}}) - 1\big)\mathbf{1}_{[0,+\infty[}(D_{0,\mathbb{R}})[G^T]_L, [G^T]_L\big\rangle_{L^2_{\mathbb{R}}} = 0, \tag{88}$$

$$\lim_{T\to+\infty}\left\langle \mathcal{K}^{\mathrm{ms}}_{\mu_0,\sigma_0}(D_{0,\mathbb{R}})\mathbf{1}_{]-\infty,0]}(D_{0,\mathbb{R}})[G^T]_L,[G^T]_L\right\rangle_{L^2_{\mathbb{R}}}=0. \tag{89}$$

Proof. We remark that

$$|\mathcal{F}([G^T]_L)(\xi)|^2=4\kappa_0B(T)|\theta(B(T)\xi)|^2, \tag{90}$$

$$\theta(B(T)\xi):=\int_{\mathbb{R}}\mathrm{e}^{-\kappa_0y}\,\mathrm{e}^{\mathrm{i}\xi B(T)\,\mathrm{e}^{-2\kappa_0y}}g(y)\,\mathrm{d}y,\qquad B(T):=C_{\kappa_0}\,\mathrm{e}^{-2\kappa_0T+\kappa_0}. \tag{91}$$

Moreover, since $G^T_2\equiv G^T_3\equiv 0$, we have for $C_1>0$

$$\begin{aligned}\Big\|\big(\mathcal{K}^{\mathrm{ms}}_{\mu_0,\nu_0}(D_{0,\mathbb{R}})-1\big)\,\mathbf{1}_{[0,+\infty[}(D_{0,\mathbb{R}})[G^T]_L\Big\|^2\\ &=C_1\int_0^{+\infty}\Big|\big(\mathcal{K}^{\mathrm{ms}}_{\mu_0,\nu_0}(\xi)-1\big)\,\mathcal{F}([G^T]_L)(\xi)\Big|^2\,\mathrm{d}\xi,\\ &=C_1\int_0^{+\infty}\left|\mathcal{K}^{\mathrm{ms}}_{\mu_0,\nu_0}\left(\frac{\eta}{B(T)}\right)-1\right|^2|\theta(\eta)|^2\,\mathrm{d}\eta.\end{aligned}$$

Since $\eta\geqslant 0$ and $\|[G^T]_L\|\leqslant\|g\|$, then $\mathcal{K}^{\mathrm{ms}}_{\mu_0,\nu_0}\big(\frac{\eta}{B(T)}\big)-1\to 0$ as $T\to+\infty$. By the Cauchy–Schwartz inequality and the Lebesgue theorem, we obtain limit (88). For limit (89), we have

$$\begin{aligned}\Big\|\mathcal{K}^{\mathrm{ms}}_{\mu_0,\nu_0}(D_{0,\mathbb{R}})\mathbf{1}_{]-\infty,0]}(D_{0,\mathbb{R}})[G^T]_L\Big\|^2&=C_2\int_{-\infty}^0\Big|\mathcal{K}^{\mathrm{ms}}_{\mu_0,\nu_0}(\xi)\mathcal{F}([G^T]_L)(\xi)\Big|^2\,\mathrm{d}\xi\\ &=C_2\int_{-\infty}^0\left|\mathcal{K}^{\mathrm{ms}}_{\mu_0,\nu_0}\left(\frac{\eta}{B(T)}\right)\right|^2|\theta(\eta)|^2\,\mathrm{d}\eta,\qquad C_2>0.\end{aligned}$$

Since $\eta\leqslant 0$ then $\mathcal{K}^{\mathrm{ms}}_{\mu_0,\nu_0}\big(\frac{\eta}{B(T)}\big)\to 0$ and we conclude as above. □

Lemma 4.3. *For $\varsigma<0(\Lambda=0)$, we have for $z\in\mathbb{C}\backslash\mathbb{R}$*

$$\left\|\big(D_{\varsigma A_\nu,]-\infty,z(0)]}\oplus D_{\varsigma A_\nu,[z(0),+\infty[}-z\big)^{-1}-(D_{\varsigma A_\nu,\mathbb{R}}-z)^{-1}\right\|\leqslant\frac{C}{|\mathrm{Im}\,z|^2},\qquad C>0. \tag{92}$$

Proof. For $f=(f_1,f_2,f_3,f_4)\in L^2_{\mathbb{R}}$ and $\mathrm{Im}\,z>0$ we have

$$((D_{0,\mathbb{R}}-z)^{-1}f)(r_*)=u(r_*),\qquad r_*\in\mathbb{R} \tag{93}$$

with

$$j=1,4\Rightarrow u_j(r_*)=-\mathrm{i}\int_{r_*}^{+\infty}\mathrm{e}^{-\mathrm{i}z(r_*-y)}f_j(y)\,\mathrm{d}y, \tag{94}$$

$$j=2,3\Rightarrow u_j(r_*)=-\mathrm{i}\int_{-\infty}^{r_*}\mathrm{e}^{\mathrm{i}z(r_*-y)}f_j(y)\,\mathrm{d}y. \tag{95}$$

At the same time, we have also

$$((D_{0,[z(0),+\infty[}-z)^{-1}f)(r_*)=u^+(r_*),\qquad r_*\in[z(0),+\infty[\tag{96}$$

with

$$u^+_1(r_*)=-\mathrm{i}\int_{r_*}^{+\infty}\mathrm{e}^{-\mathrm{i}z(r_*-y)}f_1(y)\,\mathrm{d}y,\qquad u^+_4(r_*)=-\mathrm{i}\int_{-\infty}^{r_*}\mathrm{e}^{-\mathrm{i}z(r_*-y)}f_4(y)\,\mathrm{d}y,$$

$$u^+_2(r_*)=-\mathrm{i}\int_{z(0)}^{r_*}\mathrm{e}^{\mathrm{i}z(r_*-y)}f_2(y)\,\mathrm{d}y-\mathrm{i}\,\mathrm{e}^{\mathrm{i}z(r_*-z(0))}\int_{z(0)}^{+\infty}\mathrm{e}^{\mathrm{i}zy}f_4(y)\,\mathrm{d}y,$$

$$u^+_3(r_*)=-\mathrm{i}\int_{z(0)}^{r_*}\mathrm{e}^{\mathrm{i}z(r_*-y)}f_3(y)\,\mathrm{d}y+\mathrm{i}\,\mathrm{e}^{\mathrm{i}z(r_*-z(0))}\int_{z(0)}^{+\infty}\mathrm{e}^{\mathrm{i}zy}f_1(y)\,\mathrm{d}y$$

and

$$((D_{0,]-\infty,z(0)]}-z)^{-1}f)(r_*)=u^-(r_*),\qquad r_*\in\,]-\infty,z(0)] \tag{97}$$

with

$$u_2^-(r_*)=-\mathrm{i}\int_{-\infty}^{r_*}\mathrm{e}^{\mathrm{i}z(r_*-y)}f_2(y)\,\mathrm{d}y,\qquad u_3^+(r_*)=-\mathrm{i}\int_{-\infty}^{r_*}\mathrm{e}^{\mathrm{i}z(r_*-y)}f_3(y)\,\mathrm{d}y,$$
$$u_1^-(r_*)=-\mathrm{i}\int_{r_*}^{z(0)}\mathrm{e}^{-\mathrm{i}z(r_*-y)}f_1(y)\,\mathrm{d}y-\mathrm{i}\,\mathrm{e}^{-\mathrm{i}z(r_*-z(0))}\int_{-\infty}^{z(0)}\mathrm{e}^{-\mathrm{i}zy}f_3(y)\,\mathrm{d}y,$$
$$u_4^-(r_*)=-\mathrm{i}\int_{r_*}^{z(0)}\mathrm{e}^{-\mathrm{i}z(r_*-y)}f_4(y)\,\mathrm{d}y+\mathrm{i}\,\mathrm{e}^{-\mathrm{i}z(r_*-z(0))}\int_{-\infty}^{z(0)}\mathrm{e}^{-\mathrm{i}zy}f_2(y)\,\mathrm{d}y.$$

Hence for $\operatorname{Im} z>0$ and $r_*\in\mathbb{R}$, we obtain that

$$((D_{0,]-\infty,z(0)]}\oplus D_{0,[z(0),+\infty[}-z)^{-1}f-(D_{0,\mathbb{R}}-z)^{-1}f)(r_*)=(u^-+u^+)(r_*)-u(r_*), \tag{98}$$

where

$$(u^-+u^+)(r_*)-u(r_*)=\begin{pmatrix}-\mathrm{i}\mathbf{1}_{]-\infty,z(0)]}(r_*)\,\mathrm{e}^{-\mathrm{i}z(r_*-z(0))}\int_{-\infty}^{z(0)}\mathrm{e}^{-\mathrm{i}zy}f_3(y)\,\mathrm{d}y\\ -\mathrm{i}\mathbf{1}_{[z(0),+\infty[}(r_*)\,\mathrm{e}^{\mathrm{i}z(r_*-z(0))}\int_{z(0)}^{+\infty}\mathrm{e}^{\mathrm{i}zy}f_4(y)\,\mathrm{d}y\\ \mathrm{i}\mathbf{1}_{[z(0),+\infty[}(r_*)\,\mathrm{e}^{\mathrm{i}z(r_*-z(0))}\int_{z(0)}^{+\infty}\mathrm{e}^{\mathrm{i}zy}f_1(y)\,\mathrm{d}y\\ \mathrm{i}\mathbf{1}_{]-\infty,z(0)]}(r_*)\,\mathrm{e}^{-\mathrm{i}z(r_*-z(0))}\int_{-\infty}^{z(0)}\mathrm{e}^{-\mathrm{i}zy}f_2(y)\,\mathrm{d}y\end{pmatrix}. \tag{99}$$

Moreover, since $\operatorname{Im} z>0$, by the Cauchy–Schwartz inequality we deduce that

$$\begin{aligned}j=1,4\Rightarrow\left|\int_{-\infty}^{z(0)}\mathrm{e}^{-\mathrm{i}zy}f_j(y)\,\mathrm{d}y\right|&\leqslant\frac{C_j}{\operatorname{Im} z}\|f_j\|,\\ j=2,3\Rightarrow\left|\int_{z(0)}^{+\infty}\mathrm{e}^{\mathrm{i}zy}f_j(y)\,\mathrm{d}y\right|&\leqslant\frac{C_j}{\operatorname{Im} z}\|f_j\|,\end{aligned} \tag{100}$$

with $C_j>0$. Therefore, with (98) and (99) we obtain that for $\operatorname{Im} z>0$

$$\|(D_{0,]-\infty,z(0)]}\oplus D_{0,[z(0),+\infty[}-z)^{-1}-(D_{0,\mathbb{R}}-z)^{-1}\|\leqslant\frac{C_5}{(\operatorname{Im} z)^2},\qquad C_5>0. \tag{101}$$

Obviously, we can prove the same estimate for $\operatorname{Im} z<0$ in the same way. We remark that for $\operatorname{Im} z\neq 0$

$$\begin{aligned}&\left\|\left(D_{0,]-\infty,z(0)]}\oplus D_{0,[z(0),+\infty[}-z\right)^{-1}-\left(D_{\varsigma A_\nu,]-\infty,z(0)]}\oplus D_{\varsigma A_\nu,[z(0),+\infty[}-z\right)^{-1}\right\|\\ &\quad=\left\|\left(D_{0,]-\infty,z(0)]}\oplus D_{0,[z(0),+\infty[}-z\right)^{-1}\varsigma A_\nu\left(D_{\varsigma A_\nu,]-\infty,z(0)]}\right.\right.\\ &\qquad\left.\left.\oplus D_{\varsigma A_\nu,[z(0),+\infty[}-z\right)^{-1}\right\|\leqslant\frac{C_6}{(\operatorname{Im} z)^2},\end{aligned} \tag{102}$$

with $C_6>0$ and

$$\begin{aligned}&\left\|\left(D_{\varsigma A_\nu,\mathbb{R}}-z\right)^{-1}-(D_{0,\mathbb{R}}-z)^{-1}\right\|=\left\|\left(D_{\varsigma A_\nu,\mathbb{R}}-z\right)^{-1}\varsigma A_\nu(D_{0,\mathbb{R}}-z)^{-1}\right\|\leqslant\frac{C_7}{(\operatorname{Im} z)^2},\\ &\qquad C_7>0,\end{aligned} \tag{103}$$

since ςA_ν is bounded and $\|(D-z)^{-1}\| \leqslant C|\mathrm{Im}\, z|^{-1}$, $C > 0$ with D self-adjoint on $L^2_{\mathbb{R}}$. Therefore, we obtain the result by using (101), (102), (103) and the following equality:

$$\begin{aligned}
&\left(D_{\varsigma A_\nu,]-\infty,z(0)]} \oplus D_{\varsigma A_\nu,[z(0),+\infty[} - z\right)^{-1} - \left(D_{\varsigma A_\nu,\mathbb{R}} - z\right)^{-1} \\
&\quad = \left(D_{\varsigma A_\nu,]-\infty,z(0)]} \oplus D_{\varsigma A_\nu,[z(0),+\infty[} - z\right)^{-1} - (D_{0,]-\infty,z(0)]} \oplus D_{0,[z(0),+\infty[} - z)^{-1} \\
&\qquad + (D_{0,]-\infty,z(0)]} \oplus D_{0,[z(0),+\infty[} - z)^{-1} - (D_{0,\mathbb{R}} - z)^{-1} \\
&\qquad + (D_{0,\mathbb{R}} - z)^{-1} - \left(D_{\varsigma A_\nu,\mathbb{R}} - z\right)^{-1}.
\end{aligned}$$

□

Lemma 4.4. *For $\varsigma < 0$ ($\Lambda = 0$) and $\nu \neq (2k+1)\pi$, $k \in \mathbb{R}$, the following operators are compact in L^2_0:*

$$\mathbf{1}_{[0,+\infty[}\left(D_{\varsigma A_\nu,]-\infty,z(0)]} \oplus D_{\varsigma A_\nu,[z(0),+\infty[}\right) - \mathbf{1}_{[0,+\infty[}\left(D_{\varsigma A_\nu,\mathbb{R}}\right) \tag{104}$$

$$\mathbf{1}_{]-\infty,0]}\left(D_{\varsigma A_\nu,]-\infty,z(0)]} \oplus D_{\varsigma A_\nu,[z(0),+\infty[}\right) - \mathbf{1}_{]-\infty,0]}\left(D_{\varsigma A_\nu,\mathbb{R}}\right) \tag{105}$$

$$\mathcal{K}^{\mathrm{ms}}_{1,\sigma}\left(D_{\varsigma A_\nu,]-\infty,z(0)]} \oplus D_{\varsigma A_\nu,[z(0),+\infty[}\right) - \mathcal{K}^{\mathrm{ms}}_{1,\sigma}\left(D_{\varsigma A_\nu,\mathbb{R}}\right). \tag{106}$$

Proof. To prove the result, we use the Helffer–Sjöstrand formula: given $f \in C^\infty(\mathbb{R})$ such that

$$\left|\partial_s^k f(s)\right| \leqslant C_k \langle s\rangle^{-k}, \qquad k \geqslant 0, \qquad \langle s\rangle := \sqrt{1+s^2}, \tag{107}$$

then there exists $\widetilde{f} \in C^\infty(\mathbb{C})$ with $\widetilde{f}_{|\mathbb{R}} = f$ and

$$|\partial_{\bar z}\widetilde{f}(z)| \leqslant C_N \langle \mathrm{Re}\, z\rangle^{-N-1} |\mathrm{Im}\, z|^N, \qquad C_N > 0, \tag{108}$$

$$\mathrm{supp}\widetilde{f} \subset \{z, |\mathrm{Im}\, z| \leqslant C\langle \mathrm{Re}\, z\rangle\} \tag{109}$$

such that

$$f(x) = \frac{\mathrm{i}}{2\pi}\int_{\mathbb{C}} \partial_{\bar z}\widetilde{f}(z)(x-z)^{-1}\,\mathrm{d}z \wedge \mathrm{d}\bar z. \tag{110}$$

Following [2], we can prove for $\varsigma < 0$ ($\Lambda = 0$) and $\nu \neq (2k+1)\pi$, $k \in \mathbb{R}$, that

$$\left\|D_{\varsigma A_\nu,]-\infty,z(0)]} \oplus D_{\varsigma A_\nu,[z(0),+\infty[} f\right\| \geqslant \varsigma\nu\|f\|, \quad f \in \mathcal{D}\left(D_{\varsigma A_\nu,]-\infty,z(0)]}\right) \oplus \mathcal{D}\left(D_{\varsigma A_\nu,[z(0),+\infty[}\right).$$

Therefore, if we choose $\chi \in C^\infty(\mathbb{R})$ such that

$$\varsigma\nu \leqslant t \Longrightarrow \chi(t) = 1, \qquad 0 \geqslant t \Longrightarrow \chi(t) = 0,$$

we obtain that

$$\mathbf{1}_{[0,+\infty[}\left(D_{\varsigma A_\nu,]-\infty,z(0)]} \oplus D_{\varsigma A_\nu,[z(0),+\infty[}\right) = \chi\left(D_{\varsigma A_\nu,]-\infty,z(0)]} \oplus D_{\varsigma A_\nu,[z(0),+\infty[}\right),$$
$$\mathbf{1}_{[0,+\infty[}\left(D_{\varsigma A_\nu,\mathbb{R}}\right) = \chi\left(D_{\varsigma A_\nu,\mathbb{R}}\right).$$

The function χ satisfies property (107). By using formula (110) with the spectral theorem, we have

$$\begin{aligned}
&\chi\left(D_{\varsigma A_\nu,]-\infty,z(0)]} \oplus D_{\varsigma A_\nu,[z(0),+\infty[}\right) - \chi\left(D_{\varsigma A_\nu,\mathbb{R}}\right) \\
&\quad = \frac{\mathrm{i}}{2\pi}\int_{\mathbb{C}} \partial_{\bar z}\widetilde{\chi}(z)\left[\left(D_{\varsigma A_\nu,]-\infty,z(0)]} \oplus D_{\varsigma A_\nu,[z(0),+\infty[} - z\right)^{-1}\right. \\
&\qquad \left. - \left(D_{\varsigma A_\nu,\mathbb{R}} - z\right)^{-1}\right]\mathrm{d}z \wedge \mathrm{d}\bar z.
\end{aligned} \tag{111}$$

According to estimate (108) with $N = 2$, to prove the compactness of (104) it suffices to check that

$$\left\|\left(D_{\varsigma A_\nu,]-\infty,z(0)]} \oplus D_{\varsigma A_\nu,[z(0),+\infty[} - z\right)^{-1} - \left(D_{\varsigma A_\nu,\mathbb{R}} - z\right)^{-1}\right\| \leqslant C|\mathrm{Im}\, z|^{-2}, \qquad z \in\subset \mathbb{C}\backslash\mathbb{R},$$

to obtain the norm operator convergence of (111), and the compacity in $L^2_{\mathbb{R}}$ of

$$\left(D_{\varsigma A_\nu,]-\infty,z(0)]} \oplus D_{\varsigma A_\nu,[z(0),+\infty[} - z\right)^{-1} - \left(D_{\varsigma A_\nu,\mathbb{R}} - z\right)^{-1}, \qquad z \in \mathbb{C}\backslash\mathbb{R}.$$

The first property is obvious by lemma 4.3 and the second is satisfied since the previous operator is of finite rank. The results for (105) and (106) are obtained in the same way, since for the last operators the function $\mathcal{K}^{\mathrm{ms}}_{1,\sigma} \in C^\infty(\mathbb{R})$ satisfies property (107). □

We define V_∞ thanks to V such that

$$V_\infty := \delta I_{\mathbb{R}^4} + \varsigma A_\nu = \lim_{r_*\to+\infty} V(r_*), \qquad \delta = \frac{qQ}{r_0}, \qquad \varsigma = -m\sqrt{F(r_+)}, \tag{112}$$

where A_ν as in (68).

Lemma 4.5. *Given* ${}^t(0, g_2, g_3, 0) \in C_0^\infty(\mathbb{R})^4$ *and* $\Lambda \geqslant 0$. *Then*

$$\lim_{T\to+\infty} \left\langle\left(\mathcal{K}^{\mathrm{ms}}_{1,\sigma_0}\left(D_{V_\infty,0}\right) - 1\right)\mathbf{1}_{[\delta,+\infty[}\left(D_{V_\infty,0}\right)[G^T]_L, [G^T]_L\right\rangle_{L^2_0}$$
$$= \lim_{T\to+\infty} \left\langle\left(\mathcal{K}^{\mathrm{ms}}_{\mu_0,\sigma_0}(D_{0,\mathbb{R}}) - 1\right)\mathbf{1}_{[0,+\infty[}(D_{0,\mathbb{R}})[G^T]_L, [G^T]_L\right\rangle_{L^2_{\mathbb{R}}}, \tag{113}$$

$$\lim_{T\to+\infty} \left\langle\mathcal{K}^{\mathrm{ms}}_{1,\sigma_0}\left(D_{V_\infty,0}\right)\mathbf{1}_{]-\infty,\delta]}\left(D_{V_\infty,0}\right)\mathcal{J}U_V(0,T)f, \mathcal{J}U_V(0,T)f\right\rangle_{L^2_0}$$
$$= \lim_{T\to+\infty} \left\langle\mathcal{K}^{\mathrm{ms}}_{\mu_0,\sigma_0}(D_{0,\mathbb{R}})\mathbf{1}_{]-\infty,0]}(D_{0,\mathbb{R}})[G^T]_L, [G^T]_L\right\rangle_{L^2_{\mathbb{R}}}. \tag{114}$$

Proof. If $\varsigma = 0$ $(\Lambda > 0)$, then we have clearly

$$\left\langle\left(\mathcal{K}^{\mathrm{ms}}_{1,\sigma_0}\left(D_{V_\infty,0}\right) - 1\right)\mathbf{1}_{[\delta,+\infty[}\left(D_{V_\infty,0}\right)[G^T]_L, [G^T]_L\right\rangle_{L^2_0}$$
$$= \left\langle\left(\mathcal{K}^{\mathrm{ms}}_{\mu_0,\sigma_0}(D_{0,\mathbb{R}}) - 1\right)\mathbf{1}_{[0,+\infty[}(D_{0,\mathbb{R}})[G^T]_L, [G^T]_L\right\rangle_{L^2_{\mathbb{R}}} \tag{115}$$

and

$$\left\langle\mathcal{K}^{\mathrm{ms}}_{1,\sigma_0}\left(D_{V_\infty,0}\right)\mathbf{1}_{]-\infty,\delta]}\left(D_{V_\infty,0}\right)\mathcal{J}U_V(0,T)f, \mathcal{J}U_V(0,T)f\right\rangle_{L^2_0}$$
$$= \left\langle\mathcal{K}^{\mathrm{ms}}_{\mu_0,\sigma_0}(D_{0,\mathbb{R}})\mathbf{1}_{]-\infty,0]}(D_{0,\mathbb{R}})[G^T]_L, [G^T]_L\right\rangle_{L^2_{\mathbb{R}}}. \tag{116}$$

Now, we treat the case of $\varsigma < 0$ $(\Lambda = 0)$ for the first limit. The proof for the second is obtained in the same way. By supposing that $\operatorname{supp}(g) \subset [0, R]$, $R > 0$ fixed, and $T > -\frac{1}{2\kappa_0}\ln(-z(0)) + \frac{1}{2\kappa_0}\ln\left(C_{\kappa_0}\right) + \frac{1}{2}$, we have $\operatorname{supp}(G^T) \subset]z(0), 0[$. Hence

$$\mathbf{1}_{[0,+\infty[}\left(D_{\varsigma A_\nu,]-\infty,z(0)]} \oplus D_{\varsigma A_\nu,[z(0),+\infty[}\right)[G^T]_L = 0 \oplus \mathbf{1}_{[0,+\infty[}\left(D_{\varsigma A_\nu,[z(0),+\infty[}\right)[G^T]_L, \tag{117}$$

with

$$\mathbf{1}_{[\delta,+\infty[}\left(D_{V_\infty,0}\right) = \mathbf{1}_{[0,+\infty[}\left(D_{\varsigma A_\nu,0}\right) = \mathbf{1}_{[0,+\infty[}\left(D_{\varsigma A_\nu,[z(0),+\infty[}\right) \tag{118}$$

and

$$\mathcal{K}^{\mathrm{ms}}_{1,\sigma_0}\left(D_{\varsigma A_\nu,]-\infty,z(0)]} \oplus D_{\varsigma A_\nu,[z(0),+\infty[}\right)[G^T]_L = 0 \oplus \mathcal{K}^{\mathrm{ms}}_{1,\sigma_0}\left(D_{\varsigma A_\nu,[z(0),+\infty[}\right)[G^T]_L, \tag{119}$$

with

$$\mathcal{K}^{\mathrm{ms}}_{1,\sigma_0}\left(D_{V_\infty,0}\right) = \mathcal{K}^{\mathrm{ms}}_{\mu_0,\sigma_0}\left(D_{\varsigma A_\nu,0}\right) = \mathcal{K}^{\mathrm{ms}}_{\mu_0,\sigma_0}\left(D_{\varsigma A_\nu,[z(0),+\infty[}\right). \tag{120}$$

From lemma 4.4, the following operator is compact in $L^2_{\mathbb{R}}$:

$$\mathcal{K}^{\mathrm{ms}}_{\mu_0,\sigma_0}\left(D_{\varsigma A_\nu,]-\infty,z(0)]} \oplus D_{\varsigma A_\nu,[z(0),+\infty[}\right)\mathbf{1}_{[0,+\infty[}\left(D_{\varsigma A_\nu,]-\infty,z(0)]} \oplus D_{\varsigma A_\nu,[z(0),+\infty[}\right)$$
$$- \mathcal{K}^{\mathrm{ms}}_{\mu_0,\sigma_0}\left(D_{\varsigma A_\nu,\mathbb{R}}\right)\mathbf{1}_{[0,+\infty[}\left(D_{\varsigma A_\nu,\mathbb{R}}\right).$$

By lemma VI.6 in [2]: $[G^T]_L \rightharpoonup 0$, $T \to +\infty$ in $L^2_{\mathbb{R}}$. Hence, we have the following limits:

$$\begin{aligned}\big\|0 \oplus \mathcal{K}^{\mathrm{ms}}_{\mu_0,\sigma_0}\big(D_{\varsigma A_\nu,[z(0),+\infty[}\big)\mathbf{1}_{[0,+\infty[}\big(D_{\varsigma A_\nu,[z(0),+\infty[}\big)[G^T]_L \\ - \mathcal{K}^{\mathrm{ms}}_{\mu_0,\sigma_0}\big(D_{\varsigma A_\nu,\mathbb{R}}\big)\mathbf{1}_{[0,+\infty[}\big(D_{\varsigma A_\nu,\mathbb{R}}\big)[G^T]_L\big\| \to 0, T \to +\infty \end{aligned} \tag{121}$$

and

$$\begin{aligned}\lim_{T\to+\infty}\big\langle\big(\mathcal{K}^{\mathrm{ms}}_{1,\sigma_0}\big(D_{V_\infty,0}\big) - 1\big)\mathbf{1}_{[\delta,+\infty[}\big(D_{V_\infty,0}\big)[G^T]_L, [G^T]_L\big\rangle_{L^2_0} \\ = \lim_{T\to+\infty}\big\langle\big(\mathcal{K}^{\mathrm{ms}}_{\mu_0,\sigma_0}\big(D_{\varsigma A_\nu,\mathbb{R}}\big) - 1\big)\mathbf{1}_{[0,+\infty[}\big(D_{\varsigma A_\nu,\mathbb{R}}\big)[G^T]_L, [G^T]_L\big\rangle_{L^2_{\mathbb{R}}}.\end{aligned} \tag{122}$$

First, we remark that using the Fourier transform $\mathcal{F}$:

$$\mathcal{F}\mathbf{1}_{[0,+\infty[}\big(D_{\varsigma A_\nu,\mathbb{R}}\big) = \left[\frac{1}{2} + \frac{1}{2\sqrt{\xi^2+\varsigma^2}}(\mathrm{i}\xi\Gamma^1 + \varsigma A_\nu)\right]\mathcal{F}.$$

Moreover

$$\big\|\mathcal{K}^{\mathrm{ms}}_{\mu_0,\sigma_0}\big(D_{\varsigma A_\nu,\mathbb{R}}\big)\mathbf{1}_{[0,+\infty[}\big(D_{\varsigma A_\nu,\mathbb{R}}\big)[G^T]_L - \mathcal{K}^{\mathrm{ms}}_{\mu_0,\sigma_0}(D_{0,\mathbb{R}})\mathbf{1}_{[0,+\infty[}(D_{0,\mathbb{R}})[G^T]_L\big\| \tag{123}$$

$$\begin{aligned}&\leqslant C_1\int_{\mathbb{R}}\left|\frac{\mathrm{i}\xi}{|\xi|}\Gamma^1 - \frac{1}{\sqrt{\xi^2+\varsigma^2}}(\mathrm{i}\xi\Gamma^1+\varsigma A_\nu)\right|^2 |\mathcal{F}([G^T]_L)(\xi)|^2\,\mathrm{d}\xi, \qquad C_1 > 0,\\
&\quad + C_2\int_0^{+\infty}\big|\mathcal{K}^{\mathrm{ms}}_{\mu_0,\sigma_0}(\mathrm{i}\xi\Gamma^1+\varsigma A_\nu) - \mathcal{K}^{\mathrm{ms}}_{\mu_0,\sigma_0}(\mathrm{i}\xi\Gamma^1)\big|^2|\mathcal{F}([G^T]_L)(\xi)|^2\,\mathrm{d}\xi,\\
&\quad C_2 > 0,\\
&= C_1\int_{\mathbb{R}}\left|\frac{\mathrm{i}\xi}{|\xi|}\Gamma^1 - \frac{1}{\sqrt{\xi^2+B^2(T)\varsigma^2}}(\mathrm{i}\xi\Gamma^1+B(T)\varsigma A_\nu)\right|^2 |\theta(\eta)|^2\,\mathrm{d}\eta,\\
&\quad + C_2\int_0^{+\infty}\left|\mathcal{K}^{\mathrm{ms}}_{\mu_0,\sigma_0}\left(\mathrm{i}\frac{\eta}{B(T)}\Gamma^1+\varsigma A_\nu\right) - \mathcal{K}^{\mathrm{ms}}_{\mu_0,\sigma_0}\left(\mathrm{i}\frac{\eta}{B(T)}\Gamma^1\right)\right|^2 |\theta(\eta)|^2\,\mathrm{d}\eta,\\
&= I_1 + I_2.\end{aligned} \tag{124}$$

By tedious but straightforward calculations, we obtain that

$$\mathcal{K}^{\mathrm{ms}}_{\mu_0,\sigma_0}\left(\mathrm{i}\frac{\eta}{B(T)}\Gamma^1+\varsigma A_\nu\right) - \mathcal{K}^{\mathrm{ms}}_{\mu_0,\sigma_0}\left(\mathrm{i}\frac{\eta}{B(T)}\Gamma^1\right) \longrightarrow 0, \qquad T\longrightarrow+\infty, \qquad \eta \geqslant 0. \tag{125}$$

Then, thanks to Lebesgue's theorem $\lim_{T\to+\infty} I_1 = \lim_{T\to+\infty} I_2 = 0$. We deduce that

$$\begin{aligned}\lim_{T\to+\infty}\big\langle\big(\mathcal{K}^{\mathrm{ms}}_{\mu_0,\sigma_0}\big(D_{\varsigma A_\nu,\mathbb{R}}\big) - 1\big)\mathbf{1}_{[0,+\infty[}\big(D_{\varsigma A_\nu,\mathbb{R}}\big)[G^T]_L, [G^T]_L\big\rangle_{L^2_{\mathbb{R}}} \\ = \lim_{T\to+\infty}\big\langle\big(\mathcal{K}^{\mathrm{ms}}_{\mu_0,\sigma_0}(D_{0,\mathbb{R}}) - 1\big)\mathbf{1}_{[0,+\infty[}(D_{0,\mathbb{R}})[G^T]_L, [G^T]_L\big\rangle_{L^2_{\mathbb{R}}}\end{aligned} \tag{126}$$

which entails the result. □

Lemma 4.6. *Given* $f \in C_0^\infty(\mathbb{R})^4$ *and*

$$g(t) := (W^-_{0,\mathbb{R}}f)(1-2t), \tag{127}$$

then

$$\|\mathcal{J}U_V(0,T)f - [G^{T/2}]_L\|_0 \to 0, \qquad T\to+\infty, \tag{128}$$

and

$$\mathcal{J}U_V(0,T)f \rightharpoonup 0, \qquad T\to+\infty \qquad in\ L^2_0. \tag{129}$$

Proof. This result is a consequence of lemmas 6.5, 6.7 and 6.9 of [15]. □

With this previous lemma and since all operators are uniformly bounded in L^2_0 norm, and $C_0^\infty(\mathbb{R})^4$ is dense in $L^2_{\mathbb{R}}$, we obtain easily

Lemma 4.7. *Given $f \in L^2_{\mathbb{R}}$, then for $\Lambda \geqslant 0$:*

$$\lim_{T\to+\infty} \left\langle \left(\mathcal{K}^{\mathrm{ms}}_{1,\sigma_0}\left(D_{V_\infty,0}\right) - 1\right)\mathbf{1}_{[\delta,+\infty[}\left(D_{V_\infty,0}\right)\mathcal{J}U_V(0,T)f, \mathcal{J}U_V(0,T)f\right\rangle_{L^2_0}$$
$$= \lim_{T\to+\infty} \left\langle \left(\mathcal{K}^{\mathrm{ms}}_{1,\sigma_0}\left(D_{V_\infty,0}\right) - 1\right)\mathbf{1}_{[\delta,+\infty[}\left(D_{V_\infty,0}\right)[G^{T/2}]_L, [G^{T/2}]_L\right\rangle_{L^2_0}, \tag{130}$$

$$\lim_{T\to+\infty} \left\langle \mathcal{K}^{\mathrm{ms}}_{1,\sigma_0}\left(D_{V_\infty,0}\right)\mathbf{1}_{]-\infty,\delta]}\left(D_{V_\infty,0}\right)\mathcal{J}U_V(0,T)f, \mathcal{J}U_V(0,T)f\right\rangle_{L^2_0}$$
$$= \lim_{T\to+\infty} \left\langle \mathcal{K}^{\mathrm{ms}}_{1,\sigma_0}\left(D_{V_\infty,0}\right)\mathbf{1}_{]-\infty,\delta]}\left(D_{V_\infty,0}\right)[G^{T/2}]_L, [G^{T/2}]_L\right\rangle_{L^2_0}. \tag{131}$$

Lemma 4.8. *The following operators are compact in L^2_0:*

$$\mathbf{1}_{[\delta,+\infty[}(D_{V,0}) - \mathbf{1}_{[\delta,+\infty[}\left(D_{V_\infty,0}\right), \tag{132}$$

$$\mathbf{1}_{]-\infty,\delta]}(D_{V,0}) - \mathbf{1}_{]-\infty,\delta]}\left(D_{V_\infty,0}\right), \tag{133}$$

$$\mathcal{K}^{\mathrm{ms}}_{1,\sigma}(D_{V,0}) - \mathcal{K}^{\mathrm{ms}}_{1,\sigma}\left(D_{V_\infty,0}\right). \tag{134}$$

Proof. From lemma III-10 in [2], we have the results for (132) and (133). For the last operator and as for the proof of lemma 4.4, we use the Helffer–Sjöstrand formula. We must check that

$$\left|(D_{V,0}-z)^{-1} - \left(D_{V_\infty,0}-z\right)^{-1}\right| \leqslant C|\mathrm{Im}\, z|^{-2}, \qquad z \in \mathbb{C}\backslash\mathbb{R} \tag{135}$$

and

$$(D_{V,0}-z)^{-1} - \left(D_{V_\infty,0}-z\right)^{-1} \quad \text{compact in } L^2_0 \text{ for } z \in \mathbb{C}\backslash\mathbb{R}.$$

For the second property, we remark that

$$(D_{V,0}-z)^{-1} - \left(D_{V_\infty,0}-z\right)^{-1} = (D_{V,0}-z)^{-1}(V_\infty - V)\left(D_{V_\infty,0}-z\right)^{-1} \qquad \text{for } z \in \mathbb{C}\backslash\mathbb{R}. \tag{136}$$

Moreover, $\lim_{r_*\to+\infty}(V_\infty(r_*) - V(r_*)) = 0$ and $(V_\infty - V) \in C^0(\mathbb{R})$. By the Sobolev embedding, we obtain that $\mathbf{1}_{[z(0),n]}(V_\infty - V)\left(D_{V_\infty,0}-z\right)^{-1}$ is compact in L^2_0 for all $n \in \mathbb{N}$ and $z \in \mathbb{C}\backslash\mathbb{R}$. As we have clearly

$$\left\|\mathbf{1}_{[z(0),n]}(V_\infty - V)\left(D_{V_\infty,0}-z\right)^{-1} - (V_\infty - V)\left(D_{V_\infty,0}-z\right)^{-1}\right\|_0 \to 0, \qquad \tilde{n} \to +\infty,$$

we conclude that (136) is compact in L^2_0. Finally, since $(V_\infty - V) \in L^\infty(\mathbb{R})$ and $\|(D-z)^{-1}\| \leqslant C|\mathrm{Im}\, z|^{-1}, C > 0$ with D self-adjoint on L^2_0, by (136), estimate (135) is satisfied. □

Lemma 4.9. *Given $f \in L^2_{\mathbb{R}}$, then for $\Lambda \geqslant 0$:*

$$\lim_{T\to+\infty} \left\langle \left(\mathcal{K}^{\mathrm{ms}}_{1,\sigma_0}(D_{V,0}) - 1\right)\mathbf{1}_{[\delta,+\infty[}(D_{V,0})\mathcal{J}U_V(0,T)f, \mathcal{J}U_V(0,T)f\right\rangle_{L^2_0}$$
$$= \lim_{T\to+\infty} \left\langle \left(\mathcal{K}^{\mathrm{ms}}_{1,\sigma_0}\left(D_{V_\infty,0}\right) - 1\right)\mathbf{1}_{[\delta,+\infty[}\left(D_{V_\infty,0}\right)\mathcal{J}U_V(0,T)f, \mathcal{J}U_V(0,T)f\right\rangle_{L^2_0} = 0, \tag{137}$$

$$\lim_{T\to+\infty}\left\langle \mathcal{K}^{\mathrm{ms}}_{1,\sigma_0}(D_{V,0})\mathbf{1}_{]-\infty,\delta]}(D_{V,0})\mathcal{J}U_V(0,T)f,\,\mathcal{J}U_V(0,T)f\right\rangle_{L^2_0}$$
$$=\lim_{T\to+\infty}\left\langle \mathcal{K}^{\mathrm{ms}}_{1,\sigma_0}\big(D_{V_\infty,0}\big)\mathbf{1}_{]-\infty,\delta]}\big(D_{V_\infty,0}\big)\mathcal{J}U_V(0,T)f,\,\mathcal{J}U_V(0,T)f\right\rangle_{L^2_0}=0. \tag{138}$$

Proof. For $K=\mathcal{K}^{\mathrm{ms}}_{1,\sigma_0}-1$ and $\mathbf{1}_\pm=\mathbf{1}_{[\delta,+\infty[}$ or $K=\mathcal{K}^{\mathrm{ms}}_{1,\sigma_0}$ and $\mathbf{1}_\pm=\mathbf{1}_{]-\infty,\delta]}$, we have

$$K(D_{V,0})\mathbf{1}_\pm(D_{V,0})=K(D_{V,0})\left(\mathbf{1}_\pm(D_{V,0})-\mathbf{1}_\pm\big(D_{V_\infty,0}\big)\right)$$
$$+\left(K(D_{V,0})-K\big(D_{V_\infty,0}\big)\right)\mathbf{1}_\pm\big(D_{V_\infty,0}\big)+K\big(D_{V_\infty,0}\big)\mathbf{1}_\pm\big(D_{V_\infty,0}\big).$$

We obtain the equality of the limits, by using the previous formula, lemma 4.8 and property (129). Finally, we conclude the proof of this lemma, thanks to lemmas 4.7, 4.5 and 4.2. □

Lemma 4.10. *Given $f\in L^2_{\mathbb{R}}$, then for $\Lambda\geqslant 0$:*

$$\lim_{T\to+\infty}\|\mathbf{1}_{[\delta,+\infty[}(D_{V,0})\mathcal{J}U_V(0,T)f\|_0^2=\left\langle W^-_{0,\mathbb{R}}f,\ \mathrm{e}^{\frac{2\pi}{\kappa_0}D_{0,\mathbb{R}}}\left(1+\mathrm{e}^{\frac{2\pi}{\kappa_0}D_{0,\mathbb{R}}}\right)^{-1}W^-_{0,\mathbb{R}}f\right\rangle_{L^2_{\mathbb{R}}}, \tag{139}$$

with

$$\delta=\frac{qQ}{r_0}.$$

Proof. See lemma 6.10 in [15]. □

Proposition 4.3. *Given $f\in L^2_{\mathbb{R}}$, then for $\Lambda\geqslant 0$:*

$$\lim_{T\to+\infty}\left\langle \mathcal{K}^{\mathrm{ms}}_{1,\sigma_0}(D_{V,0})\mathcal{J}U_V(0,T)f,\,\mathcal{J}U_V(0,T)f\right\rangle_{L^2_0}=\left\langle \mathcal{K}^{\mathrm{ms}}_{1,\sigma}(D_{0,\mathbb{R}})W^-_{0,\mathbb{R}}f,\,W^-_{0,\mathbb{R}}f\right\rangle_{L^2_{\mathbb{R}}}, \tag{140}$$

with

$$\sigma=\frac{2\pi}{\kappa_0}.$$

Proof. By a straightforward calculation, we have

$$\left\langle \mathcal{K}^{\mathrm{ms}}_{1,\sigma_0}(D_{V,0})\mathcal{J}U_V(0,T)f,\,\mathcal{J}U_V(0,T)f\right\rangle_{L^2_0}$$
$$=\left\langle \mathcal{K}^{\mathrm{ms}}_{1,\sigma_0}(D_{V,0})\mathbf{1}_{[\delta,+\infty[}(D_{V,0})\mathcal{J}U_V(0,T)f,\,\mathcal{J}U_V(0,T)f\right\rangle_{L^2_0}$$
$$+\left\langle \mathcal{K}^{\mathrm{ms}}_{1,\sigma_0}(D_{V,0})\mathbf{1}_{]-\infty,\delta]}(D_{V,0})\mathcal{J}U_V(0,T)f,\,\mathcal{J}U_V(0,T)f\right\rangle_{L^2_0}$$
$$=\left\langle\left(\mathcal{K}^{\mathrm{ms}}_{1,\sigma_0}(D_{V,0})-1\right)\mathbf{1}_{[\delta,+\infty[}(D_{V,0})\mathcal{J}U_V(0,T)f,\,\mathcal{J}U_V(0,T)f\right\rangle_{L^2_0}$$
$$+\|\mathbf{1}_{[\delta,+\infty[}(D_{V,0})\mathcal{J}U_V(0,T)f\|_0^2$$
$$+\left\langle \mathcal{K}^{\mathrm{ms}}_{1,\sigma_0}(D_{V,0})\mathbf{1}_{]-\infty,\delta]}(D_{V,0})\mathcal{J}U_V(0,T)f,\,\mathcal{J}U_V(0,T)f\right\rangle_{L^2_0},\qquad \delta:=\frac{qQ}{r_0}.$$

The first and the third terms are treated by lemma 9 and the second term by lemma 4.10. □

Proposition 4.4. *Given $f\in L^2_{\mathbb{R}}$, then for $\Lambda\geqslant 0$:*

$$\lim_{T\to+\infty}\left\langle \mathcal{K}^{\mathrm{ms}}_{1,\sigma_0}(D_{V,0})U_V(0,T)f,\ U_V(0,T)f\right\rangle_{L^2_0}=\left\langle \mathcal{K}^{\mathrm{ms}}_{1,\sigma_0}(D_{V,0})W^-_{V,[z(0),+\infty[}f,\,W^-_{V,[z(0),+\infty[}f\right\rangle_{L^2_0}$$
$$+\left\langle \mathcal{K}^{\mathrm{ms}}_{1,\sigma}(D_{0,\mathbb{R}})W^-_{0,\mathbb{R}}f,\,W^-_{0,\mathbb{R}}f\right\rangle_{L^2_{\mathbb{R}}}, \tag{141}$$

with

$$\sigma=\frac{2\pi}{\kappa_0}.$$

Proof. With a simple calculation, we obtain that

$$\left\langle \mathcal{K}^{\mathrm{ms}}_{1,\sigma_0}(D_{V,0})U_V(0,T)f,\, U_V(0,T)f\right\rangle_{L^2_0} = \left\langle \mathcal{K}^{\mathrm{ms}}_{1,\sigma_0}(D_{V,0})\mathcal{J}U_V(0,T)f,\, \mathcal{J}U_V(0,T)f\right\rangle_{L^2_0}$$
$$+\left\langle \mathcal{K}^{\mathrm{ms}}_{1,\sigma_0}(D_{V,0})(1-\mathcal{J})U_V(0,T)f,\, (1-\mathcal{J})U_V(0,T)f\right\rangle_{L^2_0}$$
$$+2\mathcal{R}\langle \mathbf{1}_{[\delta,+\infty[}(D_{V,0})(1-\mathcal{J})U_V(0,T)f,\, \mathbf{1}_{[\delta,+\infty[}(D_{V,0})\mathcal{J}U_V(0,T)f\rangle_{L^2_0}.$$

The last term vanishes as $T \to +\infty$ thanks to limit (129) and lemma 4.1. By lemmas 4.10 and 4.1, we conclude that the two first terms are zero as $T \to +\infty$. □

Proof of theorem 4.1. By lemma 1, the wave operator $W^-_{V_{l,\nu},[z(0),+\infty[}$ exists and is an isometry from $L^2_{\mathbb{R}}$ onto $P_{ac}(D_{V,[z(0),+\infty[})L^2_0$. Hence by using operators (65) and (63), we deduce that

$$\boldsymbol{W}^-_+ := \bigoplus_{(l,n)\in\mathcal{I}} \mathcal{E}^\nu_{ln} W^-_{V_{l,\nu},[z(0),+\infty[}\mathcal{R}^\nu_{ln}, \qquad \Lambda \geqslant 0 \tag{142}$$

exists and is an isometry from $\boldsymbol{L}^2_{\mathrm{BH}}$ onto $P_{ac}(\boldsymbol{D}_0)\boldsymbol{L}^2_0$. By definition, we have

$$\boldsymbol{\Omega}^-_{\Lambda,\to} := (\boldsymbol{W}^-_{\Lambda,\to})^*, \qquad \Lambda \geqslant 0.$$

According to the chain rule theorem, the following wave operator,

$$\boldsymbol{W}^-_{\Lambda,D} := \boldsymbol{\Omega}^-_{\Lambda,\to}(\boldsymbol{W}^-_+)^* : P_{ac}(\boldsymbol{D}_0)\boldsymbol{L}^2_0 \to \boldsymbol{L}^2_{\Lambda,\to}, \qquad \Lambda \geqslant 0, \tag{143}$$

is an isometry from $P_{ac}(\boldsymbol{D}_0)\boldsymbol{L}^2_0$ onto $\boldsymbol{L}^2_{\Lambda,\to}$. With the help of the Lebesgue theorem, proposition 4.4, the properties of operators (65), (63) and properties (62), (66) and (74), we obtain the following limit:

$$\lim_{T\to+\infty} \left\langle \mathcal{K}^{\mathrm{ms}}_{\mu_0,\sigma_0}(\boldsymbol{D}_0)\boldsymbol{U}(0,T)f,\, \boldsymbol{U}(0,T)f\right\rangle_{\mathfrak{H}}$$
$$= \lim_{T\to+\infty} \sum_{(l,n)\in\mathcal{I}} \left\langle \mathcal{K}^{\mathrm{ms}}_{\mu_0,\sigma_0}\big(D_{V_{l,\nu},0}-\delta\big)U_{V_{l,\nu}}(0,T)\mathcal{R}^\nu_{ln}f,\, U_{V_{l,\nu}}(0,T)\mathcal{R}^\nu_{ln}f\right\rangle_{L^2_0},$$
$$= \sum_{(l,n)\in\mathcal{I}} \left\langle \mathcal{K}^{\mathrm{ms}}_{1,\sigma_0}\big(D_{V_{l,\nu},0}\big)U_{V_{l,\nu}}(0,T)\mathcal{R}^\nu_{ln}f,\, U_{V_{l,\nu}}(0,T)\mathcal{R}^\nu_{ln}f\right\rangle_{L^2_0},$$
$$= \sum_{(l,n)\in\mathcal{I}} \left\langle \mathcal{K}^{\mathrm{ms}}_{1,\sigma_0}\big(D_{V_{l,\nu},0}\big)W^-_{V_{l,\nu},[z(0),+\infty[}\mathcal{R}^\nu_{ln}f,\, W^-_{V_{l,\nu},[z(0),+\infty[}\mathcal{R}^\nu_{ln}f\right\rangle_{L^2_0}$$
$$+ \sum_{(l,n)\in\mathcal{I}} \left\langle \mathcal{K}^{\mathrm{ms}}_{1,\sigma}(D_{0,\mathbb{R}})W^-_{0,\mathbb{R}}\mathcal{R}^\nu_{ln}f,\, W^-_{0,\mathbb{R}}\mathcal{R}^\nu_{ln}f\right\rangle_{L^2_{\mathbb{R}}},$$
$$=: S_1 + S_2.$$

From the definition of $\boldsymbol{W}^-_{\Lambda,D}$ and $\boldsymbol{W}^-_+$, and the intertwining properties, we deduce that for $\Lambda \geqslant 0$

$$S_1 = \sum_{(l,n)\in\mathcal{I}} \left\langle W^-_{V_{l,\nu},[z(0),+\infty[}\mathcal{K}^{\mathrm{ms}}_{1,\sigma_0}(D_{V,\mathbb{R}})\mathcal{R}^\nu_{ln}f,\; W^-_{V_{l,\nu},[z(0),+\infty[}\mathcal{R}^\nu_{ln}f\right\rangle_{L^2_0},$$
$$= \left\langle \boldsymbol{W}^-_+\mathcal{K}^{\mathrm{ms}}_{1,\sigma_0}(\boldsymbol{D}_{\mathrm{BH}}+\delta)f,\, \boldsymbol{W}^-_+f\right\rangle_{\boldsymbol{L}^2_0}$$
$$= \left\langle \boldsymbol{W}^-_{\Lambda,D}\boldsymbol{W}^-_+\mathcal{K}^{\mathrm{ms}}_{\mu_0,\sigma_0}(\boldsymbol{D}_{\mathrm{BH}})f,\, \boldsymbol{W}^-_{\Lambda,D}\boldsymbol{W}^-_+f\right\rangle_{\boldsymbol{L}^2_{\Lambda,\to}}$$
$$= \left\langle \boldsymbol{\Omega}^-_{\Lambda,\to}\mathcal{K}^{\mathrm{ms}}_{\mu_0,\sigma_0}(\boldsymbol{D}_{\mathrm{BH}})f,\, \boldsymbol{\Omega}^-_{\Lambda,\to}f\right\rangle_{\boldsymbol{L}^2_{\Lambda,\to}}$$
$$= \left\langle \mathcal{K}^{\mathrm{ms}}_{\mu_0,\sigma_0}(\boldsymbol{D}_{\Lambda,\to})\boldsymbol{\Omega}^-_{\Lambda,\to}f,\, \boldsymbol{\Omega}^-_{\Lambda,\to}f\right\rangle_{\boldsymbol{L}^2_{\Lambda,\to}}.$$

We define

$$\boldsymbol{\Omega}^-_{\leftarrow} := (\boldsymbol{W}^-_{\leftarrow})^*,$$

and remark that

$$\mathcal{P}_r \boldsymbol{D}_{\leftarrow} \mathcal{P}_r^{-1} = \bigoplus_{(l,n)\in\mathcal{I}} \mathcal{E}_{ln}^{\nu} D_{0,\mathbb{R}} \mathcal{R}_{ln}^{\nu} - \delta, \qquad \delta = \frac{qQ}{r_0}.$$

Hence, with (62) and (82), we have

$$S_2 = \left\langle \mathcal{P}_r \mathcal{K}_{\mu,\sigma}^{\mathrm{ms}}(\boldsymbol{D}_{\leftarrow})\boldsymbol{\Omega}_{\leftarrow}^{-} f, \mathcal{P}_r \boldsymbol{\Omega}_{\leftarrow}^{-} f\right\rangle_{\boldsymbol{L}^2_{\mathrm{BH}}}, \qquad \boldsymbol{L}^2_{\mathrm{BH}} = \mathcal{P}_r \boldsymbol{L}^2_{\leftarrow},$$

$$= \left\langle \mathcal{K}_{\mu,\sigma}^{\mathrm{ms}}(\boldsymbol{D}_{\leftarrow})\boldsymbol{\Omega}_{\leftarrow}^{-} f, \boldsymbol{\Omega}_{\leftarrow}^{-} f\right\rangle_{\boldsymbol{L}^2_{\leftarrow}}, \qquad \mu = \mathrm{e}^{\sigma\delta}, \qquad \sigma := \frac{2\pi}{\kappa_0}, \qquad \delta := \frac{qQ}{r_0}.$$

Therefore, we obtain limit (55). □

4.3. *Proof of theorem 3.1*

By the identity of polarization, it is sufficient to evaluate for $\Phi \in C_0^{\infty}(\mathcal{M}_{\mathrm{coll}})^4$ the following limit:

$$\lim_{T\to+\infty} \omega_{\mathcal{M}_{\mathrm{coll}}}(\boldsymbol{\Psi}^*_{\mathrm{coll}}(\Phi^T)\boldsymbol{\Psi}_{\mathrm{coll}}(\Phi^T)).$$

Since for $T > 0$ large enough, we have

$$S_{\mathrm{coll}}\Phi^T = \boldsymbol{U}(0,T)S_{\mathrm{bh}}\Phi, \qquad S_{\mathrm{bh}}\Phi := \int_{\mathbb{R}} \boldsymbol{U}(-t)\Phi(t)\,\mathrm{d}t,$$

we obtain that

$$\lim_{T\to+\infty} \omega_{\mathcal{M}_{\mathrm{coll}}}(\boldsymbol{\Psi}^*_{\mathrm{coll}}(\Phi^T)\boldsymbol{\Psi}_{\mathrm{coll}}(\Phi^T)) = \lim_{T\to+\infty}\left\langle \mathcal{K}^{\mathrm{ms}}_{\mu_0,\sigma_0}(\boldsymbol{D}_0)S_{\mathrm{coll}}\Phi^T, S_{\mathrm{coll}}\Phi^T\right\rangle_{\mathfrak{H}},$$

$$= \lim_{T\to+\infty}\left\langle \mathcal{K}^{\mathrm{ms}}_{\mu_0,\sigma_0}(\boldsymbol{D}_0)\boldsymbol{U}(0,T)S_{\mathrm{bh}}\Phi, \boldsymbol{U}(0,T)S_{\mathrm{bh}}\Phi\right\rangle_{\mathfrak{H}}. \tag{144}$$

Therefore, thanks to limit (144) of theorem 4.1, we deduce that for $\Lambda \geqslant 0$:

$$\lim_{T\to+\infty} \omega_{\mathcal{M}_{\mathrm{coll}}}(\boldsymbol{\Psi}^*_{\mathrm{coll}}(\Phi^T)\boldsymbol{\Psi}_{\mathrm{coll}}(\Phi^T)) = \left\langle \mathcal{K}^{\mathrm{ms}}_{\mu_0,\sigma_0}(\boldsymbol{D}_{\Lambda,\to})\boldsymbol{\Omega}^{-}_{\Lambda,\to}S_{\mathrm{bh}}\Phi, \boldsymbol{\Omega}^{-}_{\Lambda,\to}S_{\mathrm{bh}}\Phi\right\rangle_{\boldsymbol{L}^2_{\Lambda,\to}}$$

$$+ \left\langle \mathcal{K}^{\mathrm{ms}}_{\mu,\sigma}(\boldsymbol{D}_{\leftarrow})\boldsymbol{\Omega}^{-}_{\leftarrow}S_{\mathrm{bh}}\Phi, \boldsymbol{\Omega}^{-}_{\leftarrow}S_{\mathrm{bh}}\Phi\right\rangle_{\boldsymbol{L}^2_{\leftarrow}}$$

$$= \left\langle \mathcal{K}^{\mathrm{ms}}_{\mu_0,\sigma_0}(\boldsymbol{D}_{\Lambda,\to})S_{\Lambda,\to}\boldsymbol{\Omega}^{-}_{\Lambda,\to}\Phi, S_{\Lambda,\to}\boldsymbol{\Omega}^{-}_{\Lambda,\to}\Phi\right\rangle_{\boldsymbol{L}^2_{\Lambda,\to}}$$

$$+ \left\langle \mathcal{K}^{\mathrm{ms}}_{\mu,\sigma}(\boldsymbol{D}_{\leftarrow})S_{\leftarrow}\boldsymbol{\Omega}^{-}_{\leftarrow}\Phi, S_{\leftarrow}\boldsymbol{\Omega}^{-}_{\leftarrow}\Phi\right\rangle_{\boldsymbol{L}^2_{\leftarrow}}$$

$$= \omega^{\delta,\sigma}_{\mathrm{Haw}}(\boldsymbol{\Psi}^*_{\leftarrow}(\boldsymbol{\Omega}^{-}_{\leftarrow}\Phi)\boldsymbol{\Psi}_{\leftarrow}(\boldsymbol{\Omega}^{-}_{\leftarrow}\Phi)) + \omega^{\delta_0,\sigma_0}_{\mathrm{KMS}}(\boldsymbol{\Psi}^*_{\Lambda,\to}(\boldsymbol{\Omega}^{-}_{\Lambda,\to}\Phi)\boldsymbol{\Psi}_{\Lambda,\to}(\boldsymbol{\Omega}^{-}_{\Lambda,\to}\Phi)).$$

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